2016 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ Choose one of these courses and answer its questions only.

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, $\cdots$ in the questions represents a numeral (from 0 to 9) or the minus sign ($-$). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$ is used later in the question. In such a case, the symbol is shaded when it is used later, as A or BC.

   Note the following :

   (1) Reduce square roots ($\sqrt{\ }$) as much as possible.
   (Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

   (2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
   (Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
   $-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

   (3) If your answer to $\dfrac{\boxed{\text{A}}\sqrt{\boxed{\text{B}}}}{\boxed{\text{C}}}$ is $\dfrac{-\sqrt{3}}{4}$, mark as shown below.

   (4) If the answer to $\boxed{\text{DE}}\,x$ is $-x$, mark "$-$" for **D** and "1" for **E** as shown below.

A	●	0	1	2	3	4	5	6	7	8	9
B	⊖	0	1	2	●	4	5	6	7	8	9
C	⊖	0	1	2	3	●	5	6	7	8	9
D	●	0	1	2	3	4	5	6	7	8	9
E	⊖	0	●	2	3	4	5	6	7	8	9

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*					
Name													

Mathematics Course 1

(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label "Course 1" and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

<u>If you do not correctly fill in the appropriate oval, your answers will not be graded.</u>

I

Q 1 Consider the quadratic function in x

$$y = -\frac{1}{8}x^2 + ax + b. \qquad \cdots\cdots\cdots \quad ①$$

When we denote the coordinates of the vertex of the graph of ① by (p, q), we have

$$p = \boxed{\mathsf{A}}\,a, \qquad q = \boxed{\mathsf{B}}\,a^2 + b.$$

(1) When the vertex (p, q) is on the straight line $x + y = 1$, a and b satisfy

$$b = \boxed{\mathsf{CD}}\,a^2 - \boxed{\mathsf{E}}\,a + \boxed{\mathsf{F}},$$

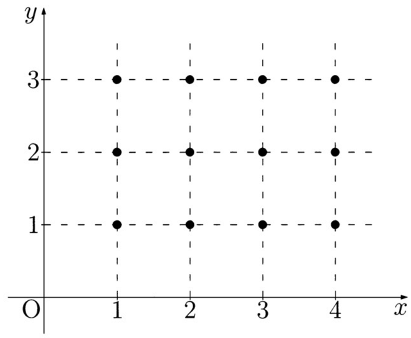
and so $8a + b$ is maximized at $a = \boxed{\mathsf{G}}$, and the maximum value is $\boxed{\mathsf{H}}$.

(2) When the graph of ① is tangent to the x-axis, the range of values of $a + b$ is

$$a + b \leqq \frac{\boxed{\mathsf{I}}}{\boxed{\mathsf{J}}}.$$

- memo -

Q 2 On a coordinate plane, 12 points are arranged as shown in the figure to the right. If we are to select three points as vertices of a triangle, how many triangles are possible in total?



First, there are **KLM** ways to select three points from the 12 points.

Next, let us find how many ways it is possible to select three or more points in a straight line.

Let us look at the two cases.

(i) There are **N** straight lines that pass through four points.

(ii) There are **O** straight lines that pass through three points.

Hence, among all combinations of three points that are in a straight line and so cannot be the vertices of a triangle, **PQ** combinations belong to case (i), and **R** combinations belong to case (ii).

Thus, the total number of possible triangles is **STU**.

In particular, if we set $(1, 1)$ as point A and $(4, 1)$ as point B, then **VW** triangles have two vertices on segment AB.

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This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{X}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

II

Q 1 We are to find the value of a such that $15x^2-2xy-8y^2-11x+22y+a$ can be factorized as the product of linear expressions in x and y.

First of all, the first three terms of the above expression form a quadratic expression in x and y that can be factorized as

$$15x^2-2xy-8y^2 = \left(\boxed{\mathsf{A}}\,x-\boxed{\mathsf{B}}\,y\right)\left(\boxed{\mathsf{C}}\,x+\boxed{\mathsf{D}}\,y\right).$$

Hence, when we set

$$\begin{aligned}&15x^2-2xy-8y^2-11x+22y+a\\&=\left(\boxed{\mathsf{A}}\,x-\boxed{\mathsf{B}}\,y+b\right)\left(\boxed{\mathsf{C}}\,x+\boxed{\mathsf{D}}\,y+c\right), \qquad \cdots\cdots\cdots \quad ①\end{aligned}$$

the right-hand side of equation ① can be expanded into

$$15x^2-2xy-8y^2+\left(\boxed{\mathsf{E}}\,b+\boxed{\mathsf{F}}\,c\right)x+\left(\boxed{\mathsf{G}}\,b-\boxed{\mathsf{H}}\,c\right)y+bc.$$

When we compare the coefficients of this expression with the coefficients of the left-hand side of equation ①, we have

$$b=\boxed{\mathsf{I}}, \quad c=-\boxed{\mathsf{J}},$$

and hence $a=-\boxed{\mathsf{KL}}$.

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Q 2 We are to find a two-digit natural number a such that $a+9$ is a multiple of 7 and $a+8$ is a multiple of 13.

First of all, $a+9$ and $a+8$ can be represented as

$$a+9 = \boxed{\mathsf{M}}\,m, \quad a+8 = \boxed{\mathsf{NO}}\,n,$$

where m and n are natural numbers. From these two equalities, we have

$$\boxed{\mathsf{M}}\,m - \boxed{\mathsf{NO}}\,n = \boxed{\mathsf{P}}. \qquad \cdots\cdots\cdots \text{①}$$

Since the pair of $m = \boxed{\mathsf{Q}}$ and $n = \boxed{\mathsf{R}}$ is an integral solution of ①, we have

$$\boxed{\mathsf{M}}\,(m - \boxed{\mathsf{Q}}) = \boxed{\mathsf{NO}}\,(n - \boxed{\mathsf{R}}). \qquad \cdots\cdots\cdots \text{②}$$

From ②, a natural number n satisfying ① can be represented as

$$n = \boxed{\mathsf{S}}\,k + \boxed{\mathsf{T}},$$

where k is an integer.

Thus

$$a = \boxed{\mathsf{UV}}\,k + \boxed{\mathsf{W}},$$

and since a is a two-digit natural number, $a = \boxed{\mathsf{XY}}$.

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This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer space $\boxed{\textbf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

Consider the two functions

$$f(x) = x^2 + 2ax + 4a - 3$$
$$g(x) = 2x + 1.$$

We are to find the condition on a for which $f(x) \geqq g(x)$ for all x and also find the range of values of the minimum of $f(x)$ under this condition.

We must find the condition under which

$$x^2 + \boxed{\mathbf{A}}\left(a - \boxed{\mathbf{B}}\right)x + \boxed{\mathbf{C}}\,a - \boxed{\mathbf{D}} \geqq 0$$

for all x.

For each of $\boxed{\mathbf{E}}$ ～ $\boxed{\mathbf{H}}$ in the following questions, choose the correct answer from among ⓪ ～ ⑦ below each question.

(1) The required condition is that a satisfy the quadratic inequality $\boxed{\mathbf{E}}$. Hence a is in the range $\boxed{\mathbf{F}}$.

⓪ $a^2 - 5a + 4 \geqq 0$　① $a^2 - 6a + 5 \geqq 0$　② $a^2 - 5a + 4 \leqq 0$

③ $a^2 - 6a + 5 \leqq 0$　④ $a \leqq 1$ or $5 \leqq a$　⑤ $1 \leqq a \leqq 5$

⑥ $1 \leqq a \leqq 4$　⑦ $a \leqq 1$ or $4 \leqq a$

(2) Let m be the minimum value of $f(x)$. Then, since $m = \boxed{\mathbf{G}}$, the range of values which m can take under the condition in (1) is $\boxed{\mathbf{H}}$.

⓪ $a^2 + 4a - 3$　① $4a^2 + 4a - 3$　② $-a^2 + 4a - 3$

③ $2a^2 - 4a + 3$　④ $-5 \leqq m \leqq 1$　⑤ $-8 \leqq m \leqq 1$

⑥ $-8 \leqq m \leqq -1$　⑦ $-5 \leqq m \leqq -1$

- memo -

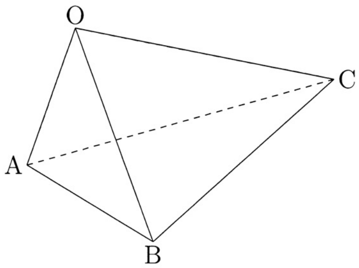
This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\textbf{I}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{III}}$ blank.

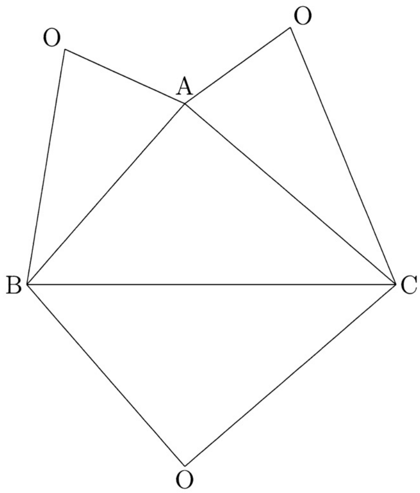
IV

The figure to the right is a net for the tetrahedron OABC. This tetrahedron satisfies

$$\mathrm{BC} = 10, \quad \mathrm{AC} = 8, \quad \sin\angle\mathrm{ACB} = \frac{3}{4},$$

$$\mathrm{OA} = 4, \quad \triangle\mathrm{ABC} \equiv \triangle\mathrm{OBC}.$$





(1) The area of the triangle ABC is $\boxed{\textbf{AB}}$.

(2) Let AH denote the perpendicular line drawn from point A to side BC. The length of AH is $\boxed{\textbf{C}}$.

(3) Let θ denote the angle formed by the plane ABC and the plane OBC. Then we have

$$\cos\theta = \frac{\boxed{\textbf{D}}}{\boxed{\textbf{E}}}, \quad \sin\theta = \frac{\boxed{\textbf{F}}\sqrt{\boxed{\textbf{G}}}}{\boxed{\textbf{H}}}.$$

(4) The volume of the tetrahedron OABC is $\dfrac{\boxed{\textbf{IJ}}\sqrt{\boxed{\textbf{K}}}}{\boxed{\textbf{L}}}$.

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This is the end of the questions for Part $\boxed{\text{IV}}$.

Leave the answer spaces $\boxed{\textbf{M}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

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< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

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and so $8a + b$ is maximized at $a = \boxed{\mathsf{G}}$, and the maximum value is $\boxed{\mathsf{H}}$.

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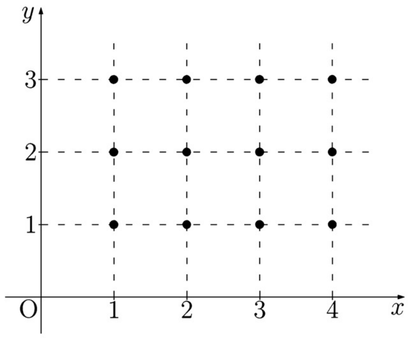

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In particular, if we set $(1, 1)$ as point A and $(4, 1)$ as point B, then $\boxed{\textbf{VW}}$ triangles have two vertices on segment AB.

- memo -

This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{X}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

II

Q 1 The triangle ABC satisfies

$$\mathrm{AB}=2, \quad \mathrm{BC}=3, \quad \mathrm{CA}=4.$$

(1) When we set $\angle\mathrm{ABC}=\theta$, the inner product $\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{BC}}$ of the vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{BC}}$ is

$$\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{BC}} = \boxed{\mathbf{AB}}\cos\theta.$$

Finding the value of $\cos\theta$ from the law of cosines, we obtain

$$\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{BC}} = \frac{\boxed{\mathbf{C}}}{\boxed{\mathbf{D}}}. \qquad \cdots\cdots\cdots \quad ①$$

(2) We divide the side BC into n equal parts by the points $\mathrm{P}_1,\ \mathrm{P}_2,\ \cdots,\ \mathrm{P}_{n-1}$ which are arranged in ascending order of the distance from B, and set $\mathrm{B}=\mathrm{P}_0$, $\mathrm{C}=\mathrm{P}_n$. We are to find the value of $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\overrightarrow{\mathrm{AP}_{k-1}}\cdot\overrightarrow{\mathrm{AP}_k}$.

When we calculate the inner product of $\overrightarrow{\mathrm{AP}_{k-1}}$ and $\overrightarrow{\mathrm{AP}_k}$ using ①, we have

$$\overrightarrow{\mathrm{AP}_{k-1}}\cdot\overrightarrow{\mathrm{AP}_k} = \boxed{\mathbf{E}} + \frac{\boxed{\mathbf{F}}\,k-\boxed{\mathbf{G}}}{2n} + \frac{\boxed{\mathbf{H}}\,(k^2-k)}{n^2}.$$

Hence we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\overrightarrow{\mathrm{AP}_{k-1}}\cdot\overrightarrow{\mathrm{AP}_k} = \frac{\boxed{\mathbf{IJ}}}{\boxed{\mathbf{K}}}.$$

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Q 2 Consider complex numbers z such that

$$z\bar{z} - (1-2i)z - (1+2i)\bar{z} \leqq 15. \qquad \cdots\cdots\cdots \quad ①$$

(1) On a complex number plane, the figure represented by inequality ① is the interior and circumference of the circle having the center $\boxed{\mathsf{L}} + \boxed{\mathsf{M}}\,i$ and the radius $\boxed{\mathsf{N}}\sqrt{\boxed{\mathsf{O}}}$.

(2) Let us consider all complex numbers z which are on the straight line

$$(1-i)z - (1+i)\bar{z} = 2i$$

and satisfy the inequality ①. Of those, let z_1 be the z such that $|z|$ is maximized and z_2 be the z such that $|z|$ is minimized. Then we have

$$z_1 = \sqrt{\boxed{\mathsf{PQ}}} + \boxed{\mathsf{R}} + \left(\sqrt{\boxed{\mathsf{ST}}} + \boxed{\mathsf{U}}\right)i,$$

$$z_2 = -\frac{\boxed{\mathsf{V}}}{\boxed{\mathsf{W}}} + \frac{\boxed{\mathsf{X}}}{\boxed{\mathsf{Y}}}\,i.$$

- memo -

This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer space $\boxed{\textbf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

Let us consider the real numbers x, y, t and u satisfying the following four conditions:

$$y \geqq |x| \qquad \cdots\cdots\cdots \quad ①$$

$$x + y = t \qquad \cdots\cdots\cdots \quad ②$$

$$x^2 + y^2 = 12 \qquad \cdots\cdots\cdots \quad ③$$

$$x^3 + y^3 = u. \qquad \cdots\cdots\cdots \quad ④$$

We are to find the ranges of values which t and u can take.

(1) From ① and ③, we see that the point (x, y) is located on the arc which is a quadrant of the circle having its center at the origin and the radius $\boxed{\mathsf{A}}\sqrt{\boxed{\mathsf{B}}}$. Moreover, the coordinates of the end points of this arc are

$$\left(\sqrt{\boxed{\mathsf{C}}},\ \sqrt{\boxed{\mathsf{D}}}\right) \quad \text{and} \quad \left(-\sqrt{\boxed{\mathsf{C}}},\ \sqrt{\boxed{\mathsf{D}}}\right).$$

From this and ②, we also see that the range of values which t can take is

$$\boxed{\mathsf{E}} \leqq t \leqq \boxed{\mathsf{F}}\sqrt{\boxed{\mathsf{G}}}. \qquad \cdots\cdots\cdots \quad ⑤$$

(2) Next, from ② and ③, we have

$$xy = \frac{\boxed{\mathsf{H}}}{\boxed{\mathsf{I}}}\left(t^2 - \boxed{\mathsf{JK}}\right)$$

and further, using ④ we also have

$$u = \frac{\boxed{\mathsf{L}}}{\boxed{\mathsf{M}}}\left(\boxed{\mathsf{NO}}\,t - t^3\right).$$

Hence, since

$$\frac{du}{dt} = \frac{\boxed{\mathsf{P}}}{\boxed{\mathsf{Q}}}\left(\boxed{\mathsf{RS}} - t^2\right),$$

the range of values which u can take under the condition ⑤ is

$$\boxed{\mathsf{T}} \leqq u \leqq \boxed{\mathsf{UV}}\sqrt{\boxed{\mathsf{W}}}.$$

- memo -

This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\textbf{X}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{III}}$ blank.

IV

Let $a > 1$. We divide the region defined by the two inequalities

$$0 \leqq x \leqq \frac{\pi}{6}, \qquad 0 \leqq y \leqq a\cos 3x$$

into two sections by the straight line $y = 1$. Let us denote the area of the section where $y \geqq 1$ by S and the area of the section where $y \leqq 1$ by T. We are to find the value of a such that $T - S$ is maximized, and also find the maximum value of $T - S$.

Let t denote the value of x $\left(0 \leqq x \leqq \frac{\pi}{6}\right)$ satisfying the equation $a\cos 3x = 1$.

Then we have

$$S = \frac{\sin 3t}{\boxed{\mathbf{A}}\cos 3t} - t,$$

$$S + T = \frac{1}{\boxed{\mathbf{B}}\cos 3t}.$$

When we set $f(t) = T - S$, we see that

$$f'(t) = \frac{\left(\boxed{\mathbf{C}} - \boxed{\mathbf{D}}\sin 3t\right)\sin 3t}{\cos^{\boxed{\mathbf{E}}} 3t}.$$

Hence $T - S$ is maximized at $t = \dfrac{\pi}{\boxed{\mathbf{FG}}}$. Thus, $T - S$ is maximized at $a = \dfrac{\boxed{\mathbf{H}}\sqrt{\boxed{\mathbf{I}}}}{\boxed{\mathbf{J}}}$,

and the maximum value is $\dfrac{\pi}{\boxed{\mathbf{K}}}$.

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This is the end of the questions for Part $\boxed{\text{IV}}$.

Leave the answer spaces $\boxed{\textbf{L}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.