2016 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such a case, the symbol is shaded when it is used later, as [A] or [BC].

Note the following :

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to [DE] x is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

(Course 2 begins on page **15**)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
(コース 1 Course 1)	コース 2 Course 2
●	○

If you do not correctly fill in the appropriate oval, your answers will not be graded.

I

Q 1 Consider the quadratic function in x

$$y = ax^2 + bx + c. \qquad \cdots\cdots\cdots \quad ①$$

The function ① takes its maximum value 16 at $x = 1$, its graph intersects the x-axis at two points, and the length of the segment connecting those two points is 8. We are to find the values of a, b and c.

From the conditions, ① can be represented as

$$y = a(x - \boxed{\mathsf{A}})^2 + \boxed{\mathsf{BC}},$$

and the coordinates of the two points at which the graph of ① and the x-axis intersect are

$$\left(-\boxed{\mathsf{D}}, 0\right), \quad \left(\boxed{\mathsf{E}}, 0\right).$$

Thus we obtain $a = \boxed{\mathsf{FG}}$. Hence we have

$$b = \boxed{\mathsf{H}}, \quad c = \boxed{\mathsf{IJ}}.$$

\- memo -

Q 2 In a box there are ten cards on which the numbers from 0 to 9 have been written successively. We take three cards out of the box using two methods and consider the probabilities.

(1) We take out three cards simultaneously.

(i) The probability that each number on the three cards is 2 or more and 6 or less is $\dfrac{\boxed{\mathsf{K}}}{\boxed{\mathsf{LM}}}$.

(ii) The probability that the smallest number is 2 or less and the greatest number is 8 or more is $\dfrac{\boxed{\mathsf{NO}}}{\boxed{\mathsf{PQ}}}$.

(2) Three times we take out one card from the box, check its number, and then return it to the box. The probability that the smallest number is 2 or more and the greatest number is 6 or less is $\dfrac{\boxed{\mathsf{R}}}{\boxed{\mathsf{S}}}$.

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This is the end of the questions for Part [I]. Leave the answer spaces **T** ∼ **Z** of Part [I] blank.

II

Q 1 Let n be a natural number and a be a real number, where $a \neq 0$. Suppose that the integral expression $x^n + y^n + z^n + a(xy + yz + zx)$ can be expressed as the product of $x + y + z$ and an integral expression P in x, y and z, i.e.

$$x^n + y^n + z^n + a(xy + yz + zx) = (x + y + z)P. \quad \cdots\cdots\cdots \quad ①$$

We are to find the values of n and a.

The equality ① holds for all x, y and z. So, consider for example, two triples of x, y and z that satisfy $x + y + z = 0$:

$$x = y = 1, \qquad z = -\boxed{\mathbf{A}}$$

and

$$x = y = -\frac{\boxed{\mathbf{B}}}{\boxed{\mathbf{C}}}, \qquad z = 1.$$

By substituting each triple in ①, we obtain the following two equations:

$$\left(-\boxed{\mathrm{A}}\right)^n = \boxed{\mathbf{D}}\,a - \boxed{\mathbf{E}} \quad \cdots\cdots\cdots \quad ②$$

$$\left(-\frac{\boxed{\mathrm{B}}}{\boxed{\mathrm{C}}}\right)^n = \frac{\boxed{\mathbf{F}}}{\boxed{\mathbf{G}}}\,a - \frac{\boxed{\mathbf{H}}}{\boxed{\mathbf{I}}}. \quad \cdots\cdots\cdots \quad ③$$

From ② and ③, we get

$$\left(\boxed{\mathrm{D}}\,a - \boxed{\mathrm{E}}\right)\left(\frac{\boxed{\mathrm{F}}}{\boxed{\mathrm{G}}}\,a - \frac{\boxed{\mathrm{H}}}{\boxed{\mathrm{I}}}\right) = \boxed{\mathbf{J}}.$$

Solving this, we obtain $a = \boxed{\mathbf{K}}$ and hence by ② that $n = \boxed{\mathbf{L}}$.

Conversely, when $a = \boxed{\mathrm{K}}$ and $n = \boxed{\mathrm{L}}$, there exists a P such that ① holds, and hence these values of a and n are the solution.

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Q 2 Consider all segments PQ of length 2 such that the end points P and Q are on the parabola $y = x^2$. Denote the mid-point of the segment PQ by M. Among all M, we are to find the coordinates of the ones nearest to the x-axis.

Let us denote the coordinates of the end points of segment PQ by $\mathrm{P}(p, p^2)$ and $\mathrm{Q}(q, q^2)$. Then the y-coordinate m of M is

$$m = \frac{p^2 + q^2}{\boxed{\mathsf{M}}}. \qquad \cdots\cdots\cdots \text{①}$$

Next, since $\mathrm{PQ} = 2$, then

$$(p-q)^2 + (p^2 - q^2)^2 = \boxed{\mathsf{N}} \qquad \cdots\cdots\cdots \text{②}$$

by the Pythagorean theorem.

Now, when we set $t = pq$, we obtain from ① and ② the quadratic equation in m

$$\boxed{\mathsf{O}}\, m^2 + m - \boxed{\mathsf{P}}\, t^2 - t - \boxed{\mathsf{Q}} = 0\,.$$

When we solve this for m, noting that $m > 0$, we have

$$m = -\frac{1}{\boxed{\mathsf{R}}} + \sqrt{\left(t + \frac{1}{\boxed{\mathsf{S}}}\right)^2 + \boxed{\mathsf{T}}}\,.$$

This shows that m is minimized when $t = -\dfrac{1}{\boxed{\mathsf{S}}}$. In this case, $pq = -\dfrac{1}{\boxed{\mathsf{S}}}$ and

$p^2 + q^2 = \dfrac{\boxed{\mathsf{U}}}{\boxed{\mathsf{V}}}$, and so we have $p + q = \pm \boxed{\mathsf{W}}$.

Thus the coordinates of the M nearest to the x-axis are $\left(\pm\dfrac{1}{\boxed{\mathsf{X}}},\ \dfrac{\boxed{\mathsf{Y}}}{\boxed{\mathsf{Z}}}\right)$.

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This is the end of the questions for Part [II].

III

(1) Answer to the following questions.

(i) Consider an integer a. When a is divided by 5, the remainder is 4. So, a can be represented as

$$a = \boxed{\textbf{A}}\,k + \boxed{\textbf{B}} \qquad (k : \text{an integer}).$$

Hence, when a^2 is divided by 5, the remainder is $\boxed{\textbf{C}}$.

(ii) The number written as the three-digit number $120_{(3)}$ in the base-3 system is $\boxed{\textbf{DE}}$ in the decimal system.

The greatest natural number that can be expressed in three digits using the base-3 system is $\boxed{\textbf{FG}}$ in the decimal system, and the smallest is $\boxed{\textbf{H}}$ in the decimal system.

(2) For each of $\boxed{\textbf{I}}$, $\boxed{\textbf{J}}$ in the following statements, choose the correct answer from among ⓪ ～ ③ below.

In the following, let a be an integer and b be a natural number.

(i) "When a is divided by 5, the remainder is 4" is $\boxed{\textbf{I}}$ for "when a^2 is divided by 5, the remainder is $\boxed{\textbf{C}}$".

(ii) "b satisfies $6 \leqq b \leqq 30$" is $\boxed{\textbf{J}}$ for "b is a three-digit number in the base-3 system".

⓪ a necessary condition but not a sufficient condition

① a sufficient condition but not a necessary condition

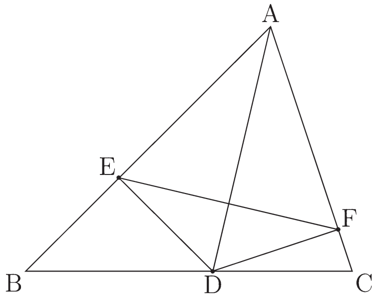
② a necessary and sufficient condition

③ neither a necessary condition nor a sufficient condition

- memo -

This is the end of the questions for Part III. Leave the answer spaces **K** ~ **Z** of Part III blank.

IV



Consider a triangle ABC where $\angle BAC = 60°$.

Let D be the point of intersection of the bisector of $\angle BAC$ and the side BC. In the figure to the right, let DE and DF be the line segments perpendicular to sides AB and AC, respectively. Let us set

$$x = \frac{AB}{AC}, \quad k = \frac{\triangle DEF}{\triangle ABC}.$$

Note that $\triangle ABC$ denotes the area of the triangle ABC, and similarly for other triangles.

(1) We are to represent k in terms of x. Since $\triangle ABD + \triangle ACD = \triangle ABC$, when we set $b = AB$, $c = AC$ and $d = AD$, we have

$$d = \frac{\sqrt{\boxed{A}}\,bc}{b+c}. \qquad \cdots\cdots\cdots ①$$

Next, since $DE = DF = \dfrac{\boxed{B}}{\boxed{C}}d$, we have

$$\triangle DEF = \frac{\sqrt{\boxed{D}}}{\boxed{EF}}d^2. \qquad \cdots\cdots\cdots ②$$

From ① and ②, we see that

$$k = \frac{d^2}{\boxed{G}\,bc} = \frac{\boxed{H}\,bc}{\boxed{I}\,(b+c)^2}.$$

Since $x = \dfrac{b}{c}$, we have

$$k = \frac{\boxed{J}\,x}{\boxed{K}\,(x+\boxed{L})^2}.$$

(2) If $BD = 8$ and $BC = 10$, then $x = \boxed{M}$ and $k = \dfrac{\boxed{N}}{\boxed{OP}}$.

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This is the end of the questions for Part IV.

Leave the answer spaces **Q** ～ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

Do not take this question booklet out of the room.

Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

<u>**If you do not correctly fill in the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Consider the quadratic function in x

$$y = ax^2 + bx + c. \qquad \cdots\cdots\cdots \quad ①$$

The function ① takes its maximum value 16 at $x = 1$, its graph intersects the x-axis at two points, and the length of the segment connecting those two points is 8. We are to find the values of a, b and c.

From the conditions, ① can be represented as

$$y = a(x - \boxed{\mathbf{A}})^2 + \boxed{\mathbf{BC}},$$

and the coordinates of the two points at which the graph of ① and the x-axis intersect are

$$\left(-\boxed{\mathbf{D}}, 0\right), \quad \left(\boxed{\mathbf{E}}, 0\right).$$

Thus we obtain $a = \boxed{\mathbf{FG}}$. Hence we have

$$b = \boxed{\mathbf{H}}, \quad c = \boxed{\mathbf{IJ}}.$$

- memo -

Q 2 In a box there are ten cards on which the numbers from 0 to 9 have been written successively. We take three cards out of the box using two methods and consider the probabilities.

(1) We take out three cards simultaneously.

(i) The probability that each number on the three cards is 2 or more and 6 or less is $\dfrac{\boxed{\mathsf{K}}}{\boxed{\mathsf{LM}}}$.

(ii) The probability that the smallest number is 2 or less and the greatest number is 8 or more is $\dfrac{\boxed{\mathsf{NO}}}{\boxed{\mathsf{PQ}}}$.

(2) Three times we take out one card from the box, check its number, and then return it to the box. The probability that the smallest number is 2 or more and the greatest number is 6 or less is $\dfrac{\boxed{\mathsf{R}}}{\boxed{\mathsf{S}}}$.

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This is the end of the questions for Part I. Leave the answer spaces T ～ Z of Part I blank.

II

Consider a sequence of positive numbers $a_1, a_2, a_3, \cdots$ which satisfies

$$a_1 = 1, \qquad a_2 = 10,$$

$$(a_n)^2 a_{n-2} = (a_{n-1})^3 \quad (n = 3, 4, \cdots). \qquad \cdots\cdots\cdots \quad ①$$

We are to find $\lim_{n\to\infty} a_n$.

By finding the common logarithm of both sides of ①, we obtain

$$\boxed{\mathsf{A}}\ \log_{10} a_n + \log_{10} a_{n-2} = \boxed{\mathsf{B}}\ \log_{10} a_{n-1}.$$

When we set $b_n = \log_{10} a_n$ $(n = 1, 2, \cdots)$, this equality is expressed as

$$\boxed{\mathsf{A}}\ b_n + b_{n-2} = \boxed{\mathsf{B}}\ b_{n-1}. \qquad \cdots\cdots\cdots \quad ②$$

By transforming ②, we have

$$b_n - b_{n-1} = \frac{1}{\boxed{\mathsf{C}}}(b_{n-1} - b_{n-2}) \quad (n = 3, 4, \cdots),$$

which gives

$$b_n - b_{n-1} = \left(\frac{1}{\boxed{\mathsf{C}}}\right)^{n-\boxed{\mathsf{D}}}(b_2 - b_1) \quad (n = 2, 3, \cdots). \qquad \cdots\cdots\cdots \quad ③$$

(This question is continued on the next page.)

Since $b_1 = \boxed{\mathsf{E}}$ and $b_2 = \boxed{\mathsf{F}}$, from ③ we get

$$b_n = \sum_{k=2}^{n} \left(\frac{1}{\boxed{\mathsf{C}}} \right)^{k-\boxed{\mathsf{G}}}$$

and hence

$$b_n = \boxed{\mathsf{H}} - \left(\frac{1}{\boxed{\mathsf{C}}} \right)^{n-\boxed{\mathsf{I}}}.$$

Finally, we obtain

$$\lim_{n \to \infty} a_n = \boxed{\mathsf{JKL}}.$$

This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer spaces $\boxed{\mathsf{M}}$ ~ $\boxed{\mathsf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

Let α and β be the two solutions of the quadratic equation $x^2+\sqrt{3}x+1=0$, where $0<\arg\alpha<\arg\beta<2\pi$. Consider the complex numbers z satisfying the following three conditions:

$$\begin{cases} \arg\dfrac{\alpha-z}{\beta-z}=\dfrac{\pi}{2} & \cdots\cdots\cdots \quad ① \\ (1+i)z+(1-i)\overline{z}+k=0 & \cdots\cdots\cdots \quad ② \\ \dfrac{\pi}{2}<\arg z<\pi, & \cdots\cdots\cdots \quad ③ \end{cases}$$

where k is a real number.

Let us denote the points on the complex number plane which express α, β and z by A, B and P.

(1) The arguments of α and β are

$$\arg\alpha=\frac{\boxed{\mathbf{A}}}{\boxed{\mathbf{B}}}\pi \quad \text{and} \quad \arg\beta=\frac{\boxed{\mathbf{C}}}{\boxed{\mathbf{D}}}\pi.$$

(2) For each of $\boxed{\mathbf{E}}$ ～ $\boxed{\mathbf{Q}}$ in the following sentences, choose the correct answer from among ⓪ ～ ⑨ below.

Since $\boxed{\mathbf{E}}=\dfrac{\pi}{2}$ from ①, the point P is located on the circumference of the circle with the center $-\dfrac{\sqrt{\boxed{\mathbf{F}}}}{\boxed{\mathbf{G}}}$ and the radius $\dfrac{\boxed{\mathbf{H}}}{\boxed{\mathbf{I}}}$.

On the other hand, from ②, the point P is on the straight line which has the slope $\boxed{\mathbf{J}}$ and intersects the imaginary axis at $\dfrac{\boxed{\mathbf{K}}}{\boxed{\mathbf{L}}}ki$.

From these, we see that when n is the number of complex numbers z which simultaneously satisfy ①, ② and ③, the maximum value of n is $\boxed{\mathbf{M}}$, and in this case the range of values of k is

$$\boxed{\mathbf{N}}+\sqrt{\boxed{\mathbf{O}}}<k<\sqrt{\boxed{\mathbf{P}}}+\sqrt{\boxed{\mathbf{Q}}},$$

where $\boxed{\mathbf{P}}<\boxed{\mathbf{Q}}$.

⓪ 0	① 1	② 2	③ 3	④ 4
⑤ 5	⑥ 6	⑦ ∠PAB	⑧ ∠PBA	⑨ ∠APB

- memo -

This is the end of the questions for Part III. Leave the answer spaces **R** ～ **Z** of Part III blank.

IV

Q 1 Let x satisfy the inequality

$$2\left(\log_{\frac{1}{3}} x\right)^2 + 9\log_{\frac{1}{3}} x + 9 \leqq 0 . \quad \cdots\cdots\cdots \quad ①$$

We are to find the maximum value of the function

$$f(x) = (\log_3 x)\left(\log_3 \frac{x}{3}\right)\left(\log_3 \frac{x}{9}\right) . \quad \cdots\cdots\cdots \quad ②$$

The range of values of x satisfying ① is

$$\boxed{\mathsf{A}}\sqrt{\boxed{\mathsf{B}}} \leqq x \leqq \boxed{\mathsf{CD}} .$$

When we set $t = \log_3 x$, the range of values of t is

$$\frac{\boxed{\mathsf{E}}}{\boxed{\mathsf{F}}} \leqq t \leqq \boxed{\mathsf{G}} .$$

When we express the right side of ② in terms of t and consider it as a function $g(t)$, its derivative is

$$g'(t) = \boxed{\mathsf{H}}\, t^2 - \boxed{\mathsf{I}}\, t + \boxed{\mathsf{J}} .$$

Hence $f(x)$ is maximized at $x = \boxed{\mathsf{KL}}$, and its maximum value is $\boxed{\mathsf{M}}$.

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Q 2 Let $a > 0$. Consider the region of a plane bounded by the curve $y = \sqrt{x}e^{-x}$, the x-axis, and the straight line $x = a$ which passes through the point $\mathrm{A}(a, 0)$, and let V be the volume of the solid obtained by rotating this region once about the x-axis.

(1) V is expressed as a function in a by

$$V = -\frac{\pi}{4}\left\{\left(\boxed{\mathsf{N}}\,a + \boxed{\mathsf{O}}\right)e^{-\boxed{\mathsf{P}}\,a} - \boxed{\mathsf{Q}}\right\}.$$

(2) Suppose that the point A starts at the origin and moves along the x-axis in the positive direction and that its speed at t seconds is $4t$. Then the rate of change of V at t seconds is

$$\frac{dV}{dt} = \boxed{\mathsf{R}}\,\pi t^{\boxed{\mathsf{S}}}\, e^{-\boxed{\mathsf{T}}\,t^{\boxed{\mathsf{U}}}}.$$

This rate of change is maximized at

$$t = \frac{\sqrt{\boxed{\mathsf{V}}}}{4},$$

and the value of V at this time is

$$V = -\frac{\pi}{8}\left(\boxed{\mathsf{W}}\,e^{-\frac{\boxed{\mathsf{X}}}{\boxed{\mathsf{Y}}}} - \boxed{\mathsf{Z}}\right).$$

- memo -

This is the end of the questions for Part IV.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.