2018 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ Choose one of these courses and answer its questions only.

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign(−). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet(mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such a case, the symbol is shaded when it is used later, as [A] or [BC].

Note the following :

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to [DE] x is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label "Course 1" and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

<u>If you do not correctly fill in the appropriate oval, your answers will not be graded.</u>

I

Q 1 Let us consider the quadratic function

$$f(x) = \frac{1}{4}x^2 - (2a-1)x + a,$$

where a is a real number.

(1) The coordinates of the vertex of the graph of $y = f(x)$ are

$$\left(\boxed{\mathsf{A}}\,a - \boxed{\mathsf{B}},\ -\boxed{\mathsf{C}}\,a^2 + \boxed{\mathsf{D}}\,a - \boxed{\mathsf{E}} \right).$$

(2) The range of a such that the graph of $y = f(x)$ and the x-axis intersect at two different points, A and B, is

$$a < \frac{\boxed{\mathsf{F}}}{\boxed{\mathsf{G}}} \quad \text{or} \quad \boxed{\mathsf{H}} < a.$$

(3) The range of a such that the x-coordinates of both points A and B in (2) are greater than or equal to 0 and less than or equal to 6 is

$$\boxed{\mathsf{I}} < a \leqq \frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{LM}}}.$$

- memo -

Q 2 We have four cards of different sizes. We are to paint each card either red, black, blue or yellow. However, we may paint more than one card with the same color.

(1) There are a total of **NOP** ways of painting the cards.

(2) There are **QR** ways of painting them using all four colors.

(3) There are **ST** ways of painting two cards red, one card black, and one card blue.

(4) There are **UVW** ways of painting the four cards using three colors.

(5) There are **XY** ways of painting them using two colors.

- memo -

This is the end of the questions for Part [I]. Leave the answer space [Z] of Part [I] blank.

II

Q 1 We are to find the positive number a satisfying $a^3 = 9 + \sqrt{80}$.

Let us consider the positive number b which satisfies $b^3 = 9 - \sqrt{80}$.

Then

$$\begin{cases} a^3 + b^3 = \boxed{\text{AB}} & \cdots\cdots\cdots \text{①} \\ ab = \boxed{\text{C}} & \cdots\cdots\cdots \text{②} \end{cases}$$

holds.

First, using ②, ① can be transformed into

$$(a+b)^3 - \boxed{\text{D}}(a+b) = \boxed{\text{AB}}.$$

Then, setting $x = a + b$, we have

$$x^3 - \boxed{\text{D}}x = \boxed{\text{AB}}.$$

Transforming this equation, we obtain

$$x^3 - 27 = \boxed{\text{D}}\left(x - \boxed{\text{E}}\right),$$

which gives

$$\left(x - \boxed{\text{F}}\right)\left(x^2 + \boxed{\text{G}}x + \boxed{\text{H}}\right) = 0.$$

From that we have $x = \boxed{\text{I}}$ and hence

$$a + b = \boxed{\text{I}}. \quad \cdots\cdots\cdots \text{③}$$

Thus, from ②, ③ and $a > b$, we have

$$a = \frac{\boxed{\text{J}} + \sqrt{\boxed{\text{K}}}}{\boxed{\text{L}}}.$$

- memo -

Q 2 Let a be a constant other than 0. Let

$$f(x) = x^2 + 2ax - 4a - 12,$$
$$g(x) = ax^2 + 2x - 4a + 4.$$

(1) When the solutions of $f(x) = 0$ and the solutions of $g(x) = 0$ coincide, a is **MN**, and their solutions are $x =$ **OP** and $x =$ **Q**.

(2) $g(x) = 0$ has just one solution when $a = \dfrac{\boxed{\text{R}}}{\boxed{\text{S}}}$, and in this case the solution is $x =$ **TU**.

(3) The range of a such that $f(x) < g(x)$ for all x is

V $\leqq a <$ **WX**.

- memo -

This is the end of the questions for Part II. Leave the answer spaces **Y**, **Z** of Part II blank.

III

Let n be a two-digit natural number such that the remainder of n^3 divided by 66 is n. We are to find the number of such n's and to find the prime numbers among them.

From the conditions we have

$$n^3 = \boxed{\mathbf{AB}}\,p + n \quad (0 < n \leqq \boxed{\mathbf{CD}}),$$

where p is the integer quotient of n^3 divided by 66. This can be transformed into

$$n(n-1)(n+1) = \boxed{\mathrm{AB}}\,p.$$

Since either $n-1$ or n has to be a multiple of $\boxed{\mathbf{E}}$ and either $n-1$, n or $n+1$ has to be a multiple of $\boxed{\mathbf{F}}$, and furthermore $\boxed{\mathrm{E}}$ and $\boxed{\mathrm{F}}$ are mutually prime, we know that $n(n-1)(n+1)$ is a multiple of $\boxed{\mathbf{G}}$. (Write the answers in the order $1 < \boxed{\mathrm{E}} < \boxed{\mathrm{F}} < \boxed{\mathrm{G}}$.) Hence one of $n-1$, n and $n+1$ must be a multiple of $\boxed{\mathbf{HI}}$.

So, since $n \leqq \boxed{\mathrm{CD}}$, the number of n's where $n-1$ is a multiple of $\boxed{\mathrm{HI}}$ is $\boxed{\mathbf{J}}$, where n is a multiple of $\boxed{\mathrm{HI}}$ is $\boxed{\mathbf{K}}$, and where $n+1$ is a multiple of $\boxed{\mathrm{HI}}$ is $\boxed{\mathbf{L}}$.

Thus, the number of n's is $\boxed{\mathbf{MN}}$ and the prime numbers among them are $\boxed{\mathbf{OP}}$, $\boxed{\mathbf{QR}}$, $\boxed{\mathbf{ST}}$, in ascending order.

- memo -

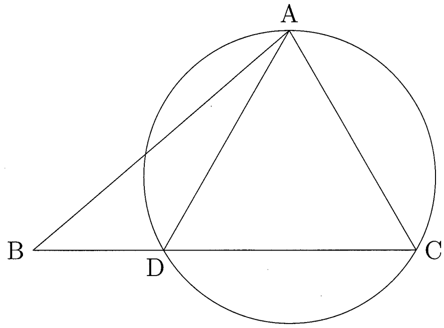
This is the end of the questions for Part III. Leave the answer spaces **U** ~ **Z** of Part III blank.

IV

The triangle ABC in the right figure satisfies

$$\mathrm{AB} = 4, \quad \mathrm{AC} = 3 \quad \text{and} \quad \angle \mathrm{B} = 30^\circ.$$

D is the point on side BC such that AC = AD. Let us consider the circumscribed circle O of triangle ACD.



(1) Since $\sin B = \dfrac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}$, we have $\sin C = \dfrac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}}$.

Hence the radius of circle O is $\dfrac{\boxed{\mathsf{E}}}{\boxed{\mathsf{F}}}$.

(2) We have

$$\mathrm{BC} = \boxed{\mathsf{G}}\sqrt{\boxed{\mathsf{H}}} + \sqrt{\boxed{\mathsf{I}}}$$

and

$$\mathrm{BD} = \boxed{\mathsf{J}}\sqrt{\boxed{\mathsf{K}}} - \sqrt{\boxed{\mathsf{L}}}.$$

Let us denote the intersection of side AB and circle O by E. Then

$$\mathrm{BE} = \frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}}.$$

Hence the relationships between the areas of triangles BDE, ADE and ACD are

$$\triangle\mathrm{BDE} : \triangle\mathrm{ADE} = \boxed{\mathsf{O}} : \boxed{\mathsf{P}},$$

$$\triangle\mathrm{BDE} : \triangle\mathrm{ACD} = \boxed{\mathsf{Q}}\left(\boxed{\mathsf{J}}\sqrt{\boxed{\mathsf{K}}} - \sqrt{\boxed{\mathsf{L}}}\right) : \boxed{\mathsf{RS}}\sqrt{\boxed{\mathsf{T}}}.$$

- memo -

This is the end of the questions for Part IV.

Leave the answer spaces **U** ～ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

Do not take this question booklet out of the room.

Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label "Course 2" and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

<u>**If you do not correctly fill in the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Let us consider the quadratic function

$$f(x) = \frac{1}{4}x^2 - (2a-1)x + a,$$

where a is a real number.

(1) The coordinates of the vertex of the graph of $y = f(x)$ are

$$\left(\boxed{\mathbf{A}}\,a - \boxed{\mathbf{B}},\ -\boxed{\mathbf{C}}\,a^2 + \boxed{\mathbf{D}}\,a - \boxed{\mathbf{E}}\right).$$

(2) The range of a such that the graph of $y = f(x)$ and the x-axis intersect at two different points, A and B, is

$$a < \frac{\boxed{\mathbf{F}}}{\boxed{\mathbf{G}}} \quad \text{or} \quad \boxed{\mathbf{H}} < a.$$

(3) The range of a such that the x-coordinates of both points A and B in (2) are greater than or equal to 0 and less than or equal to 6 is

$$\boxed{\mathbf{I}} < a \leqq \frac{\boxed{\mathbf{JK}}}{\boxed{\mathbf{LM}}}.$$

- memo -

Q 2 We have four cards of different sizes. We are to paint each card either red, black, blue or yellow. However, we may paint more than one card with the same color.

(1) There are a total of **NOP** ways of painting the cards.

(2) There are **QR** ways of painting them using all four colors.

(3) There are **ST** ways of painting two cards red, one card black, and one card blue.

(4) There are **UVW** ways of painting the four cards using three colors.

(5) There are **XY** ways of painting them using two colors.

- memo -

This is the end of the questions for Part I. Leave the answer space Z of Part I blank.

II

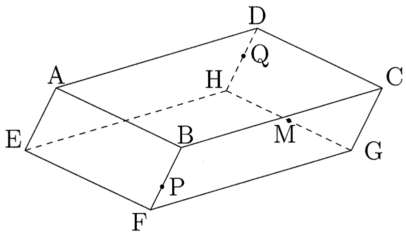
Q 1 The parallelepiped in the right figure satisfies

$AB = 2, \quad AD = 3, \quad AE = 1,$

$\angle BAD = 60^\circ, \quad \angle BAE = 90^\circ, \quad \angle DAE = 120^\circ,$

and M is the midpoint of edge GH.

Let us take points P and Q on edges BF and DH, respectively, such that the four points A, P, M, and Q are on the same plane. We are to find the points P and Q which maximize the length of the line segment PQ.



(1) Setting $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AD}$ and $\vec{c} = \overrightarrow{AE}$, the inner products of these vectors are

$$\vec{a} \cdot \vec{b} = \boxed{\mathsf{A}}, \quad \vec{b} \cdot \vec{c} = -\frac{\boxed{\mathsf{B}}}{\boxed{\mathsf{C}}}, \quad \vec{c} \cdot \vec{a} = \boxed{\mathsf{D}}.$$

(2) Let s and t satisfy $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$, and set $BP:PF = s:(1-s)$, $DQ:QH = t:(1-t)$. Since the four points A, P, M and Q are on the same plane, there exist two real numbers α and β such that

$$\overrightarrow{AM} = \alpha\overrightarrow{AP} + \beta\overrightarrow{AQ}.$$

Hence s and t satisfy

$$s = \boxed{\mathsf{E}}\left(\boxed{\mathsf{F}} - t\right).$$

Then $|\overrightarrow{PQ}|$ may be expressed in terms of t as

$$|\overrightarrow{PQ}|^2 = \boxed{\mathsf{G}}\,t^2 - \boxed{\mathsf{HI}}\,t + \boxed{\mathsf{JK}}.$$

Hence the length of segment PQ is maximized when $\boxed{\mathsf{L}}$. Here, for $\boxed{\mathsf{L}}$ choose the correct answer from among choices ⓪ ~ ⑤ below.

⓪ $s = 0, \quad t = 1$ ① $s = 0, \quad t = \frac{1}{2}$ ② $s = \frac{1}{2}, \quad t = \frac{3}{4}$

③ $s = \frac{2}{3}, \quad t = \frac{2}{3}$ ④ $s = 1, \quad t = \frac{1}{2}$ ⑤ $s = 1, \quad t = \frac{2}{3}$

\- memo -

Q 2 For any x and y satisfying $x > 0$ and $y > 0$, let m be the smallest value among $\frac{y}{x}$, x and $\frac{8}{y}$.

Also, let A be the set of points (x, y) where $m = \frac{y}{x}$, and let B be the set of points (x, y) where $m = \frac{8}{y}$.

(1) For **M** ~ **S** in the following sentence, choose the correct answer from among choices ⓪ ~ ⑦ below.

A and B can be expressed as follows:

$$A = \left\{(x, y) \;\middle|\; \boxed{\text{M}} \leqq \boxed{\text{N}},\ \ \boxed{\text{O}} \leqq 8\boxed{\text{P}}\right\},$$

$$B = \left\{(x, y) \;\middle|\; 8\boxed{\text{Q}} \leqq \boxed{\text{R}},\ \ 8 \leqq \boxed{\text{S}}\right\}.$$

⓪ x	① y	② $x+y$	③ $x-y$
④ x^2	⑤ xy	⑥ y^2	⑦ x^2+y^2

(2) For **T** and **U** in the following sentence, choose the correct answer from among choices ⓪ ~ ⑧ on the right page.

When sets A and B are indicated on the xy-plane, A is the shaded portion of graph **T** and B is the shaded portion of graph **U**. Note that the x and y axes are not included in the shaded portions.

(3) We are to find the maximum value of m when a point $\text{P}(x, y)$ moves within $A \cup B$.

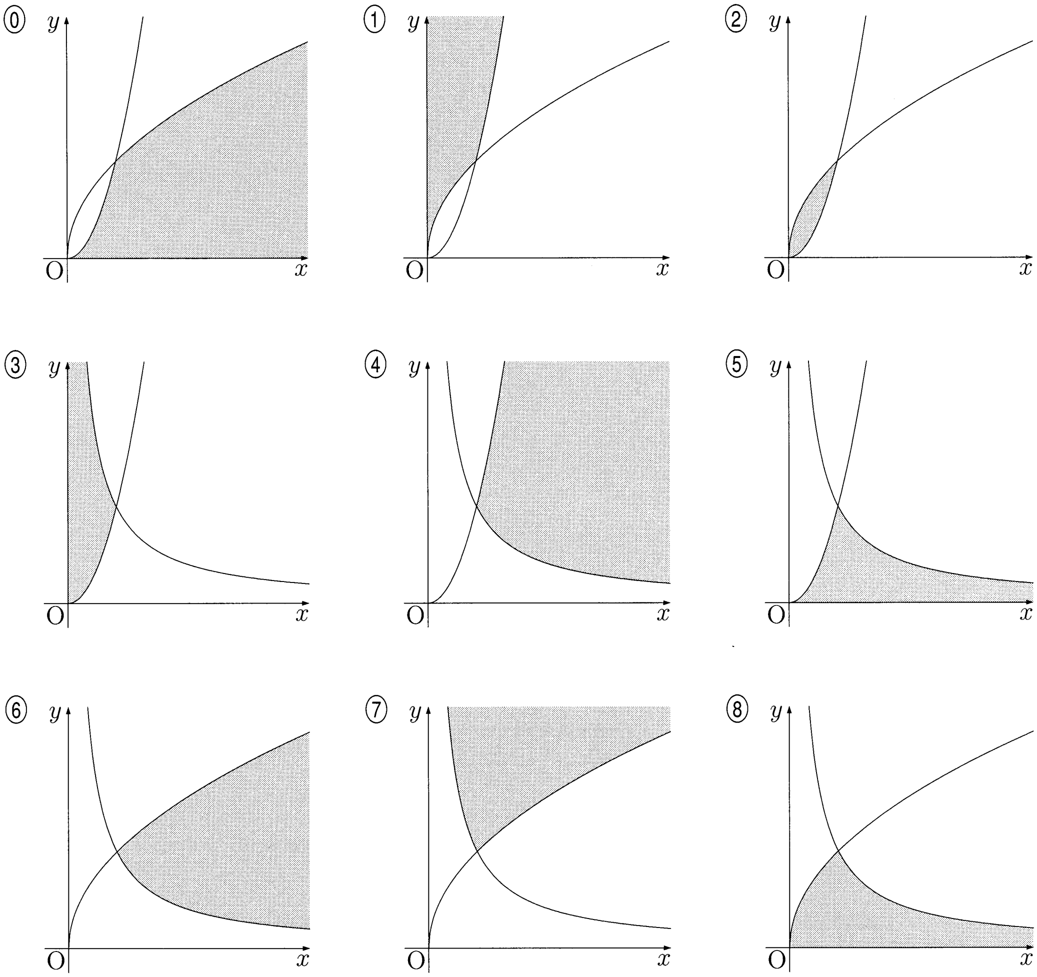
When $\text{P}(x, y) \in A$, since $y = mx$, we need to find the point P which maximizes the slope of the straight line passing through the origin O and P.

Also, when $\text{P}(x, y) \in B$, since $m = \frac{8}{y}$, we need to find the point P at which the y coordinate of P is minimized.

From the above, at $(x, y) = \left(\boxed{\text{V}}, \boxed{\text{W}}\right)$, m takes the maximum value $\boxed{\text{X}}$.

(This question is continued on the next page.)

[The choices for (2)]



This is the end of the questions for Part II. Leave the answer spaces **Y**, **Z** of Part II blank.

III

We are to find the maximum and the minimum values of the function

$$f(x) = 4\sin^3 x + 4\cos^3 x - 8\sin 2x - 7,$$

where $0 \leqq x \leqq \pi$.

Set $t = \sin x + \cos x$. Since

$$\sin x + \cos x = \sqrt{\boxed{\mathsf{A}}}\sin\left(x + \frac{\boxed{\mathsf{B}}}{\boxed{\mathsf{C}}}\pi\right), \quad \text{(note: have } \boxed{\mathsf{B}} < \boxed{\mathsf{C}}\text{)}$$

the range of values which t takes is $-\boxed{\mathsf{D}} \leqq t \leqq \sqrt{\boxed{\mathsf{E}}}$.

Next, since

$$\sin 2x = t^2 - \boxed{\mathsf{F}}$$

and

$$4\sin^3 x + 4\cos^3 x = -\boxed{\mathsf{G}}\,t^3 + \boxed{\mathsf{H}}\,t,$$

we have

$$f(x) = -\boxed{\mathsf{G}}\,t^3 - \boxed{\mathsf{I}}\,t^2 + \boxed{\mathsf{H}}\,t + \boxed{\mathsf{J}}. \quad \cdots\cdots\cdots \text{①}$$

When we set the right side of ① as $g(t)$ and differentiate with respect to t, we have

$$g'(t) = -\boxed{\mathsf{K}}\left(\boxed{\mathsf{L}}\,t - \boxed{\mathsf{M}}\right)\left(t + \boxed{\mathsf{N}}\right).$$

Hence at $t = \dfrac{\boxed{\mathsf{O}}}{\boxed{\mathsf{P}}}$, $g(t)\ \left(= f(x)\right)$ takes the maximum value $\dfrac{\boxed{\mathsf{QR}}}{\boxed{\mathsf{ST}}}$, and at $t = \sqrt{\boxed{\mathsf{U}}}$, it takes the minimum value $\boxed{\mathsf{V}}\sqrt{\boxed{\mathsf{W}}} - \boxed{\mathsf{XY}}$.

- memo -

This is the end of the questions for Part [III]. Leave the answer space [Z] of Part [III] blank.

IV

Let

$$a_n = \int_0^1 x^{2n}\sqrt{1-x^2}\ dx \quad (n = 0, 1, 2, \cdots).$$

We are to find the value of the limit $\lim_{n\to\infty} \dfrac{a_n}{a_{n-1}}$.

(1) First, let us find a_0 and a_1. Since the area of a circle with the radius 1 is π, we see that

$$a_0 = \int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi}{\boxed{\text{A}}}.$$

Next, the partial integral method applied to a_1 gives

$$\begin{aligned} a_1 &= \int_0^1 x^2\sqrt{1-x^2}\ dx \\ &= -\frac{\boxed{\text{B}}}{\boxed{\text{C}}}\left[x(1-x^2)^{\frac{\boxed{\text{D}}}{\boxed{\text{E}}}}\right]_0^1 + \frac{\boxed{\text{F}}}{\boxed{\text{G}}}\int_0^1 (1-x^2)^{\frac{\boxed{\text{H}}}{\boxed{\text{I}}}}\ dx \\ &= \frac{\boxed{\text{J}}}{\boxed{\text{K}}}\left\{\int_0^1 \sqrt{1-x^2}\ dx - \int_0^1 x^{\boxed{\text{L}}}\sqrt{1-x^2}\ dx\right\}. \end{aligned}$$

Thus we have

$$a_1 = \frac{\pi}{\boxed{\text{MN}}}.$$

(This question is continued on the next page.)

(2) For **O** ~ **U** in the following sentences, choose the correct answer from among choices ⓪ ~ ⑨ below.

When the partial integral method is applied to a_n in the same way as for a_1, we get

$$a_n = \frac{\boxed{\text{O}}}{\boxed{\text{P}}}\left\{\int_0^1 x^{\boxed{\text{Q}}}\sqrt{1-x^2}\,dx - \int_0^1 x^{\boxed{\text{R}}}\sqrt{1-x^2}\,dx\right\} \quad (n = 1, 2, 3, \cdots).$$

Hence we have

$$\left(\boxed{\text{S}}\right) a_n = \left(\boxed{\text{T}}\right) a_{n-1},$$

and so

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \boxed{\text{U}}.$$

⓪ 0 ① 1 ② 2 ③ 3 ④ 4

⑤ $2n-2$ ⑥ $2n-1$ ⑦ $2n$ ⑧ $2n+1$ ⑨ $2n+2$

This is the end of the questions for Part IV.

Leave the answer spaces **V** ~ **Z** of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.