Question Booklet

2012 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instruction for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instruction for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). Completely fill in your answer for each letter in the corresponding line of the answer sheet (mark-sheet).

Note the following :

(1) Express square roots ($\sqrt{\ }$) in their simplest form.
(Example : Substitute $2\sqrt{3}$ for $\sqrt{12}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to $\boxed{DE}\,x$ is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

3. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*					
Name													

Mathematics Course 1

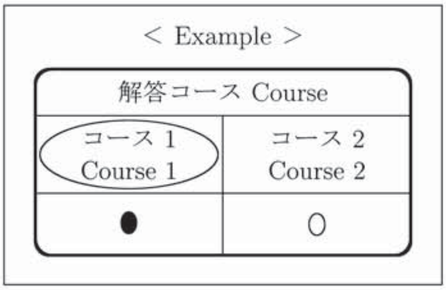
(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely black out the oval under the label on your answer sheet as shown in the example on the right.



If you do not correctly black out the appropriate oval, your answers will not be graded.

I

Q 1 Let a and b be constants where $a > 0$. Translate the graph of the quadratic function

$$y = 4x^2 + 2ax + b$$

by a in the x-direction and by $1 - 7a$ in the y-direction. If this graph passes through the point $(0, 4)$, we have

$$b = \boxed{\text{AB}}\,a^2 + \boxed{\text{C}}\,a + \boxed{\text{D}},$$

and the quadratic function representing the graph resulting from these translations is

$$y = \boxed{\text{E}}\,x^2 - \boxed{\text{F}}\,ax + \boxed{\text{G}}. \qquad \cdots\cdots\cdots \text{①}$$

When the graph of quadratic function ① is tangent to the x-axis, we have $a = \dfrac{\boxed{\text{H}}}{\boxed{\text{I}}}$, and the x-coordinate of the point of tangency is $x = \boxed{\text{J}}$.

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Q 2 Consider the polynomial

$$P = x^2 + 2(a-1)x - 8a - 8.$$

(1) Let a be a rational number. If the value of P is a rational number when $x = 1 - \sqrt{2}$, then $a =$ **K** and in this case the value of P is $P =$ **LM**.

(2) Let x and a be positive integers. We are to investigate x and a which are such that the value of P is a prime number.

When we factorize P, we have

$$P = (x - \boxed{\text{N}})(x + \boxed{\text{O}}\,a + \boxed{\text{P}}).$$

Hence x must be **Q**.

Furthermore, the smallest possible a is **R**, and in this case the value of P is $P =$ **ST**.

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This is the end of the questions for Part I. Leave the answer spaces U ～ Z of Part I blank.

II

Q 1 Let P be a point in a plane with a coordinate system that is initially located at the origin $(0, 0)$ and moves in the plane according to the following rule:

One dice is thrown. When the number on the dice is a multiple of three, point P moves 1 unit in the positive direction of the x-axis, and when the number on the dice is not a multiple of three, point P moves 1 unit in the positive direction of the y-axis.

Assume that the dice is thrown four times.

(1) The probability that P reaches point $(3, 1)$ is $\dfrac{\boxed{\text{A}}}{\boxed{\text{BC}}}$.

(2) Altogether, the number of the points which P can reach is $\boxed{\text{D}}$, and the coordinates of these points can be expressed in terms of an integer k as

$$\left(k,\ \boxed{\text{E}} - k\right) \quad \left(\boxed{\text{F}} \leqq k \leqq \boxed{\text{G}}\right).$$

Let us denote the probability that P can reach a given point $\left(k,\ \boxed{\text{E}} - k\right)$ by p_k. Then the maximum value of p_k is $\dfrac{\boxed{\text{HI}}}{\boxed{\text{BC}}}$, and the minimum value of p_k is $\dfrac{\boxed{\text{J}}}{\boxed{\text{BC}}}$.

(3) The probability that P passes through point $(1, 1)$ and reaches point $(2, 2)$ is $\dfrac{\boxed{\text{KL}}}{\boxed{\text{BC}}}$.

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Q 2 Let D, E and F be the three points which divide internally the three sides AB, BC and CA, respectively, of a triangle ABC in the ratio of $k : (1-k)$, where $0 < k \leqq \frac{1}{2}$.

(1) When $k = \frac{1}{3}$, we are to find how many times greater the area of triangle ABC is than the area of triangle DEF. Since

$$\triangle \mathrm{ADF} = \triangle \mathrm{BED} = \triangle \mathrm{CFE} = \frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}} \triangle \mathrm{ABC},$$

it follows that

$$\triangle \mathrm{ABC} = \boxed{\mathsf{O}} \triangle \mathrm{DEF}.$$

(2) The area of the triangle DEF is half of the area of the triangle ABC when

$$k(1-k) = \frac{\boxed{\mathsf{P}}}{\boxed{\mathsf{Q}}},$$

that is, when

$$k = \frac{\boxed{\mathsf{R}} - \sqrt{\boxed{\mathsf{S}}}}{\boxed{\mathsf{T}}}.$$

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This is the end of the questions for Part II. Leave the answer spaces U ~ Z of Part II blank.

III

Let m be a real number. On a plane with the coordinate system, in which the origin is denoted by O, consider the parabola $y = x^2$ and the two points on it,

$$\mathrm{A}(a,\, ma+1), \quad \mathrm{B}(b,\, mb+1) \quad (a < 0 < b).$$

(1) The x-coordinates a and b of the two points A and B can be expressed in terms of m as

$$a = \frac{m - \sqrt{D}}{\boxed{\mathbf{A}}}, \quad b = \frac{m + \sqrt{D}}{\boxed{\mathbf{B}}},$$

where the expression D is

$$D = m^2 + \boxed{\mathbf{C}}.$$

(2) Let the coordinates of the point of intersection of the segment AB and the y-axis be denoted by $(0, c)$. Then $c = \boxed{\mathbf{D}}$.

(3) Further, when the area S of the triangle OAB with the three vertices O, A and B is expressed in terms of a and b, we have

$$S = \frac{1}{2}\boxed{\mathbf{E}},$$

where $\boxed{\mathrm{E}}$ is the appropriate choice from among ⓪ ～ ⑤.

⓪ $a+b$ ① $a-b$ ② $b-a$ ③ a^2+b^2 ④ a^2-b^2 ⑤ b^2-a^2

Also, when S is represented in terms of m, we have

$$S = \frac{\boxed{\mathbf{F}}}{\boxed{\mathbf{G}}}\sqrt{m^2 + \boxed{\mathbf{H}}}.$$

Hence the value of S is minimalized when $m = \boxed{\mathbf{I}}$, and its minimum value is $S = \boxed{\mathbf{J}}$.

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This is the end of the questions for Part III. Leave the answer spaces K ～ Z of Part III blank.

IV

Let a be a real number. Consider the quadratic expressions in x

$$A = x^2 + ax + 1$$
$$B = x^2 + (a+3)x + 4.$$

(1) The range of values taken by a such that there exists a real number x satisfying $A+B=0$ is

$$a \leqq -\sqrt{\boxed{\mathsf{AB}}} - \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}} \quad \text{or} \quad \sqrt{\boxed{\mathsf{AB}}} - \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}} \leqq a.$$

(2) The range of values taken by a such that there exists a real number x satisfying $AB=0$ is

$$a \leqq \boxed{\mathsf{EF}} \quad \text{or} \quad \boxed{\mathsf{G}} \leqq a.$$

(3) There exists a real number x satisfying $A^2+B^2=0$ only when $a = \boxed{\mathsf{H}}$. In this case $x = \boxed{\mathsf{IJ}}$.

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This is the end of the questions for Part IV.

Leave the answer spaces K ～ Z of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

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< Example >

If you do not correctly black out the appropriate oval, your answers will not be graded.

I

Q 1 Let a and b be constants where $a > 0$. Translate the graph of the quadratic function

$$y = 4x^2 + 2ax + b$$

by a in the x-direction and by $1 - 7a$ in the y-direction. If this graph passes through the point $(0, 4)$, we have

$$b = \boxed{\mathsf{AB}}\,a^2 + \boxed{\mathsf{C}}\,a + \boxed{\mathsf{D}},$$

and the quadratic function representing the graph resulting from these translations is

$$y = \boxed{\mathsf{E}}\,x^2 - \boxed{\mathsf{F}}\,ax + \boxed{\mathsf{G}}. \qquad \cdots\cdots\cdots \quad ①$$

When the graph of quadratic function ① is tangent to the x-axis, we have $a = \dfrac{\boxed{\mathsf{H}}}{\boxed{\mathsf{I}}}$, and the x-coordinate of the point of tangency is $x = \boxed{\mathsf{J}}$.

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Q 2 Consider the polynomial

$$P = x^2 + 2(a-1)x - 8a - 8.$$

(1) Let a be a rational number. If the value of P is a rational number when $x = 1 - \sqrt{2}$, then $a =$ **K** and in this case the value of P is $P =$ **LM** .

(2) Let x and a be positive integers. We are to investigate x and a which are such that the value of P is a prime number.

When we factorize P, we have

$$P = (x - \boxed{\mathbf{N}})(x + \boxed{\mathbf{O}}\,a + \boxed{\mathbf{P}}).$$

Hence x must be **Q** .

Furthermore, the smallest possible a is **R** , and in this case the value of P is $P =$ **ST** .

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This is the end of the questions for Part I. Leave the answer spaces U ~ Z of Part I blank.

II

Consider a sequence $\{a_n\}\,(n = 1, 2, 3, \cdots)$ where the sum of the first n terms is

$$\sum_{k=1}^{n} a_k = n^2 + 3n.$$

(1) Then $a_n = \boxed{\mathsf{A}}\,n + \boxed{\mathsf{B}}$.

(2) For the sequence $\{b_n\}\,(n = 1, 2, 3, \cdots)$, where $b_n = n^2 - 5n - 6$, the number of terms satisfying $b_n < 0$ is $\boxed{\mathsf{C}}$, and the sum of such terms is $-\boxed{\mathsf{DE}}$.

(3) It follows that for the sequences $\{a_n\}$ and $\{b_n\}$ in (1) and (2),

$$\sum_{k=1}^{n} \frac{k^2\, b_k}{a_k} = \frac{1}{\boxed{\mathsf{F}}}\, n\,(n + \boxed{\mathsf{G}})(n^2 - \boxed{\mathsf{H}}\,n - \boxed{\mathsf{I}}).$$

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This is the end of the questions for Part II. Leave the answer spaces **J** ~ **Z** of Part II blank.

III

Let a, b and c be positive real numbers. Consider a triangle ABC whose vertices are the three points A$(a, 0)$, B$(3, b)$ and C$(0, c)$ on a plane with the coordinate system. Assume that the circumscribed circle of the triangle ABC passes through the origin O$(0, 0)$ and that $\angle\text{BAC} = 60°$.

(1) Since $\angle\text{AOB} = \boxed{\textbf{AB}}°$, we obtain $b = \sqrt{\boxed{\textbf{C}}}$.

(2) The equation of the circumscribed circle is

$$\left(x - \frac{a}{\boxed{\textbf{D}}}\right)^2 + \left(y - \frac{c}{\boxed{\textbf{E}}}\right)^2 = \frac{a^2 + c^2}{\boxed{\textbf{F}}},$$

and c can be expressed in terms of a as $c = \sqrt{\boxed{\textbf{G}}}\left(\boxed{\textbf{H}} - a\right)$.

(3) Let D denote the point of intersection of the segment OB and the segment AC. Set $\alpha = \angle\text{OAC}$ and $\beta = \angle\text{ADB}$. When $a = 2\sqrt{3}$, it follows that

$$\tan\alpha = \boxed{\textbf{I}} - \sqrt{\boxed{\textbf{J}}}, \quad \tan\beta = \boxed{\textbf{K}}.$$

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This is the end of the questions for Part III. Leave the answer spaces L ～ Z of Part III blank.

IV

Q 1 Let a be a positive real number. We are to investigate local extrema of the function

$$f(x) = x^2 - 5 + 4a\log(2x + a + 8) \qquad \left(-\frac{a}{2} - 4 < x < -2\right).$$

(1) When we differentiate the function $f(x)$ with respect to x, we obtain

$$f'(x) = \frac{\boxed{\mathsf{A}}\left(\boxed{\mathsf{B}}x + a\right)\left(x + \boxed{\mathsf{C}}\right)}{\boxed{\mathsf{D}}x + a + \boxed{\mathsf{E}}}.$$

(2) Since a condition of a is that $a > 0$ and the domain of $f(x)$ is $-\frac{a}{2} - 4 < x < -2$, the range of values of a such that $f(x)$ has both a local maximum and a local minimum is

$$\boxed{\mathsf{F}} < a < \boxed{\mathsf{G}} \quad \text{or} \quad \boxed{\mathsf{G}} < a.$$

In such a case, the sum of the local maximum and the local minimum is

$$\frac{a^2}{\boxed{\mathsf{H}}} + \boxed{\mathsf{I}} + \boxed{\mathsf{J}}\,a\log\boxed{\mathsf{K}}\,a.$$

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Q 2 For a positive integer n and a real number a, consider the function

$$f_n(a) = \int_0^{\pi} (\cos x + a \sin 2nx)^2 \, dx.$$

(1) When we transform $f_n(a)$ into

$$f_n(a) = \int_0^{\pi} \left\{ \frac{1 + \cos \boxed{\mathsf{L}} x}{2} + a^2 \frac{1 - \cos \boxed{\mathsf{M}} nx}{2} + a\Big(\sin(2n+1)x + \sin(2n-1)x\Big) \right\} dx$$

and calculate the definite integral on the right side, we obtain

$$f_n(a) = \frac{\pi}{\boxed{\mathsf{N}}} a^2 + \frac{\boxed{\mathsf{O}}\, n}{\boxed{\mathsf{P}}\, n^2 - \boxed{\mathsf{Q}}} a + \frac{\pi}{\boxed{\mathsf{R}}}.$$

(2) Let a_n denote the value of a at which $f_n(a)$ is minimalized, and set $S_N = \sum_{n=1}^{N} \frac{a_n}{n}$.

Then

$$S_N = -\frac{\boxed{\mathsf{S}}}{\pi} \sum_{n=1}^{N} \left(\frac{1}{2n - \boxed{\mathsf{T}}} - \frac{1}{2n + \boxed{\mathsf{U}}} \right)$$

$$= -\frac{\boxed{\mathsf{S}}}{\pi} \left(\boxed{\mathsf{V}} - \frac{1}{\boxed{\mathsf{W}}\, N + \boxed{\mathsf{X}}} \right).$$

Hence we obtain

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \lim_{N \to \infty} S_N = -\frac{\boxed{\mathsf{Y}}}{\pi}.$$

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This is the end of the questions for Part IV. Leave the answer space **Z** of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

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