Question Booklet

2012 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign(−). Completely black out your answer for each letter in the corresponding line of the answer sheet(mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such as a case, the symbol is shaded when it is used later, as [A] or [BC].

   Note the following :

   (1)Reduce square roots ($\sqrt{\ }$) as much as possible.
   (Example : Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

   (2)For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
   (Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
   $-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

   (3)If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

   (4)If the answer to [DE] x is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
(コース 1 Course 1)	コース 2 Course 2
●	○

<u>If you do not correctly black out the appropriate oval, your answers will not be graded.</u>

I

Q 1 Let $a \neq 0$. Let G be a curve which is symmetric with respect to the origin $(0, 0)$ to the graph of the quadratic function in x

$$y = ax^2 - 4x - 4a. \qquad \cdots\cdots\cdots \quad ①$$

(1) The coordinates of the vertex of the graph of ① are

$$\left(\frac{\boxed{\mathsf{A}}}{a},\ -\frac{\boxed{\mathsf{B}}}{a} - 4a \right).$$

(2) Among the following choices, the quadratic function whose graph is G is $\boxed{\mathsf{C}}$.

⓪ $y = ax^2 + 4x + 4a$ ① $y = ax^2 + 4x - 4a$ ② $y = ax^2 - 4x + 4a$

③ $y = -ax^2 + 4x + 4a$ ④ $y = -ax^2 - 4x + 4a$ ⑤ $y = -ax^2 - 4x - 4a$

(3) The curve G intersects the graph of the quadratic function ① at the two points

$$\left(\boxed{\mathsf{DE}},\ \boxed{\mathsf{F}} \right) \text{ and } \left(\boxed{\mathsf{G}},\ \boxed{\mathsf{HI}} \right).$$

(4) Let $a = 2$. Then over the interval $\boxed{\mathsf{DE}} \leqq x \leqq \boxed{\mathsf{G}}$, the maximum and the minimum values of the quadratic function whose graph is G are $\boxed{\mathsf{JK}}$ and $\boxed{\mathsf{LM}}$, respectively.

- memo -

Q 2 Consider the following equation in x

$$|ax - 11| = 4x - 10, \qquad \cdots\cdots\cdots \quad ①$$

where a is a constant.

(1) Equation ① can be rewritten without using the absolute value symbol as

when $ax \geqq 11$, then $\left(a - \boxed{\text{N}}\right)x = \boxed{\text{O}}$;

when $ax < 11$, then $\left(a + \boxed{\text{P}}\right)x = \boxed{\text{QR}}$.

(2) When $a = \sqrt{7}$, the solution of equation ① is

$$x = \frac{\boxed{\text{S}}\left(\boxed{\text{T}} - \sqrt{\boxed{\text{U}}}\right)}{\boxed{\text{V}}}.$$

(3) Let a be a positive integer. When equation ① has a positive integral solution, we have $a = \boxed{\text{W}}$, and that solution $x = \boxed{\text{X}}$.

- memo -

This is the end of the questions for Part I. Leave the answer spaces **Y**, **Z** of Part I blank.

II

Q 1 There are two boxes, A and B.

In box A, there are three cards on which the number 0 is written, two cards on which the number 2 is written, and one card on which the number 3 is written.

In box B, there are two cards on which the number 1 is written, and three cards on which the number 2 is written.

Take two cards together from box A and one card from box B. Denote the product of the numbers on the three cards by X.

The total number of values which X can take is **A**. The maximum value which X can take is **BC**. The minimum value which X can take is **D**.

The probability that $X =$ BC is $\frac{\mathbf{E}}{\mathbf{FG}}$, and the probability that $X =$ D is $\frac{\mathbf{H}}{\mathbf{I}}$.

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Q 2 Consider a triangle ABC where AB = 8, AC = 5 and ∠BAC = 60°. Take point D and point E on sides AB and AC, respectively, such that the segment DE divides triangle ABC into two parts with the equal areas. Set $x = \mathrm{AD}$.

(1) When we represent AE in terms of x, we have $\mathrm{AE} = \dfrac{\boxed{\mathsf{JK}}}{x}$.

(2) When E moves along side AC, the range of the values which x can take is

$$\boxed{\mathsf{L}} \leqq x \leqq \boxed{\mathsf{M}}.$$

(3) Since $\mathrm{DE}^2 = \left(x - \dfrac{\boxed{\mathsf{NO}}}{x}\right)^2 + \boxed{\mathsf{PQ}}$, the length of segment DE is minimized at $x = \boxed{\mathsf{R}}\sqrt{\boxed{\mathsf{S}}}$, and its length there is $\boxed{\mathsf{T}}\sqrt{\boxed{\mathsf{U}}}$.

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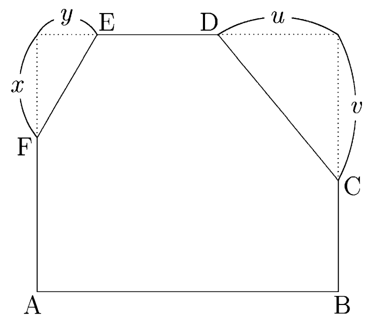
This is the end of the questions for Part II. Leave the answer spaces **V** ~ **Z** of Part II blank.

III

Consider a figure made by cuting two corners from a rectangle, as in the diagram to the right. The lengths of the sides are

$$\mathrm{AB}=11,\quad \mathrm{BC}=4,\quad \mathrm{CD}=2\sqrt{13},$$
$$\mathrm{DE}=5,\quad \mathrm{EF}=2\sqrt{5},\quad \mathrm{FA}=6.$$

We are to find the area of this figure.



First, extend the sides of the figure as in the diagram and denote the sides forming the right angles by x, y, u and v. Then

$$u = \boxed{\mathbf{A}} - y, \qquad v = x + \boxed{\mathbf{B}}.$$

Substituting these expressions in the equation $u^2 + v^2 = \boxed{\mathbf{CD}}$ and also using the equation $x^2 + y^2 = \boxed{\mathbf{EF}}$, we obtain

$$x = \boxed{\mathbf{G}}\,y - \boxed{\mathbf{H}}.$$

Then, since

$$\boxed{\mathbf{I}}\,y^2 - \boxed{\mathbf{J}}\,y - \boxed{\mathbf{K}} = 0,$$

we obtain $y = \boxed{\mathbf{L}}$.

From this we have $x = \boxed{\mathbf{M}}$, and further $u = \boxed{\mathbf{N}}$ and $v = \boxed{\mathbf{O}}$. Finally we conclude that the area of this figure is $\boxed{\mathbf{PQ}}$.

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This is the end of the questions for Part III. Leave the answer spaces **R** ~ **Z** of Part III blank.

IV

Let x and y be real numbers which satisfy

$$3x^2 + 2xy + 3y^2 = 32. \quad \cdots\cdots\cdots \quad ①$$

Then we are to find the ranges of the values of $x + y$ and xy.

First, we set

$$a = x + y. \quad \cdots\cdots\cdots \quad ②$$

By eliminating y from ① and ②, we obtain the quadratic equation in x

$$\boxed{\mathbf{A}}\,x^2 - \boxed{\mathbf{B}}\,ax + \boxed{\mathbf{C}}\,a^2 - 32 = 0.$$

Since x is a real number, we have

$$\boxed{\mathbf{DE}} \leqq a \leqq \boxed{\mathbf{F}}. \quad \cdots\cdots\cdots \quad ③$$

Next, we set

$$b = xy. \quad \cdots\cdots\cdots \quad ④$$

From ①, ② and ④ we obtain

$$b = \frac{\boxed{\mathbf{G}}}{\boxed{\mathbf{H}}}\,a^2 - \boxed{\mathbf{I}}. \quad \cdots\cdots\cdots \quad ⑤$$

Hence from ③ and ⑤ we have

$$\boxed{\mathbf{JK}} \leqq b \leqq \boxed{\mathbf{L}}.$$

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This is the end of the questions for Part IV.

Leave the answer spaces **M** ~ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

Do not take this question booklet out of the room.

Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	(コース 2 Course 2)
○	●

<u>**If you do not correctly black out the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Let $a \neq 0$. Let G be a curve which is symmetric with respect to the origin $(0, 0)$ to the graph of the quadratic function in x

$$y = ax^2 - 4x - 4a. \quad \cdots\cdots\cdots \quad ①$$

(1) The coordinates of the vertex of the graph of ① are

$$\left(\frac{\boxed{\mathsf{A}}}{a},\ -\frac{\boxed{\mathsf{B}}}{a} - 4a \right).$$

(2) Among the following choices, the quadratic function whose graph is G is $\boxed{\mathsf{C}}$.

⓪ $y = ax^2 + 4x + 4a$ ① $y = ax^2 + 4x - 4a$ ② $y = ax^2 - 4x + 4a$

③ $y = -ax^2 + 4x + 4a$ ④ $y = -ax^2 - 4x + 4a$ ⑤ $y = -ax^2 - 4x - 4a$

(3) The curve G intersects the graph of the quadratic function ① at the two points

$$\left(\boxed{\mathsf{DE}}, \boxed{\mathsf{F}} \right) \text{ and } \left(\boxed{\mathsf{G}}, \boxed{\mathsf{HI}} \right).$$

(4) Let $a = 2$. Then over the interval $\boxed{\mathsf{DE}} \leqq x \leqq \boxed{\mathsf{G}}$, the maximum and the minimum values of the quadratic function whose graph is G are $\boxed{\mathsf{JK}}$ and $\boxed{\mathsf{LM}}$, respectively.

- memo -

Q 2 Consider the following equation in x

$$|ax - 11| = 4x - 10, \qquad \cdots\cdots\cdots \quad ①$$

where a is a constant.

(1) Equation ① can be rewritten without using the absolute value symbol as

when $ax \geqq 11$, then $\left(a - \boxed{\mathsf{N}}\right)x = \boxed{\mathsf{O}}$;

when $ax < 11$, then $\left(a + \boxed{\mathsf{P}}\right)x = \boxed{\mathsf{QR}}$.

(2) When $a = \sqrt{7}$, the solution of equation ① is

$$x = \frac{\boxed{\mathsf{S}}\left(\boxed{\mathsf{T}} - \sqrt{\boxed{\mathsf{U}}}\right)}{\boxed{\mathsf{V}}}.$$

(3) Let a be a positive integer. When equation ① has a positive integral solution, we have $a = \boxed{\mathsf{W}}$, and that solution $x = \boxed{\mathsf{X}}$.

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This is the end of the questions for Part I. Leave the answer spaces **Y**, **Z** of Part I blank.

II

Suppose that a triangle ABC which is inscribed in a circle O of radius 2 satisfies

$$3\overrightarrow{OA} + 4\overrightarrow{OB} + 2\overrightarrow{OC} = \vec{0}. \qquad \cdots\cdots\cdots \quad ①$$

Let D denote the point of intersection of the straight line AO and the segment BC. We are to find the lengths of the segments AD and BD.

(1) When we set $\overrightarrow{OD} = k\,\overrightarrow{OA}$ where k is a real number, we have

$$\overrightarrow{OD} = -\frac{\boxed{\mathrm{A}}}{\boxed{\mathrm{B}}}\,k\,\overrightarrow{OB} - \frac{\boxed{\mathrm{C}}}{\boxed{\mathrm{D}}}\,k\,\overrightarrow{OC}.$$

As the three points B, C and D are located on a straight line, we obtain $k = \dfrac{\boxed{\mathrm{EF}}}{\boxed{\mathrm{G}}}$.

From this we derive that $\mathrm{OD} = \boxed{\mathrm{H}}$ and finally obtain

$$\mathrm{AD} = \boxed{\mathrm{I}}.$$

(2) From (1) we see that $\mathrm{BD} = \dfrac{\boxed{\mathrm{J}}}{\boxed{\mathrm{K}}}\mathrm{BC}$. So in order to find the length of the segment BD, we should find the length of the segment BC.

First we note that

$$\mathrm{BC}^2 = \boxed{\mathrm{L}} - \boxed{\mathrm{M}}\,\overrightarrow{OB}\cdot\overrightarrow{OC},$$

where $\overrightarrow{OB}\cdot\overrightarrow{OC}$ represents the inner product of $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Since we know from ① that $\left|4\overrightarrow{OB} + 2\overrightarrow{OC}\right|^2 = \boxed{\mathrm{NO}}$, we have

$$\overrightarrow{OB}\cdot\overrightarrow{OC} = \frac{\boxed{\mathrm{PQR}}}{\boxed{\mathrm{S}}}.$$

Hence we obtain $\mathrm{BC} = \dfrac{\boxed{\mathrm{T}}\sqrt{\boxed{\mathrm{U}}}}{\boxed{\mathrm{V}}}$ and finally from that

$$\mathrm{BD} = \frac{\sqrt{\boxed{\mathrm{W}}}}{\boxed{\mathrm{X}}}.$$

- memo -

This is the end of the questions for Part II. Leave the answer spaces **Y**, **Z** of Part II blank.

III

Let x and y be positive numbers which satisfy

$$(\log_2 x)^2 + (\log_2 y)^2 = \log_2 \frac{8x^2}{y^2}. \qquad \cdots\cdots\cdots \quad ①$$

We are to find the maximum value of xy^2 and the values of x and y at that point.

(1) The right side of ① can be transformed into

$$\log_2 \frac{8x^2}{y^2} = \boxed{\mathsf{A}} \log_2 x - \boxed{\mathsf{B}} \log_2 y + \boxed{\mathsf{C}}.$$

So, setting $X = \log_2 x$ and $Y = \log_2 y$, we can express ① in terms of X and Y as

$$\left(X - \boxed{\mathsf{D}}\right)^2 + \left(Y + \boxed{\mathsf{E}}\right)^2 = \boxed{\mathsf{F}}. \qquad \cdots\cdots\cdots \quad ②$$

(2) Set $k = \log_2 xy^2$. Using X and Y above, this equality can be transformed into

$$X + \boxed{\mathsf{G}} Y - k = 0. \qquad \cdots\cdots\cdots \quad ③$$

If we graph ② and ③ on a plane with coordinates (X, Y), the graph of ② is a circle, and the graph of ③ is a straight line. When k is maximized, the graph of ③ is tangent to the graph of ②. Hence, when $k = \boxed{\mathsf{H}}$, xy^2 takes the maximum value $\boxed{\mathsf{IJ}}$ and in this case $x = \boxed{\mathsf{K}}$ and $y = \boxed{\mathsf{L}}$.

- memo -

This is the end of the questions for Part III. Leave the answer spaces **M** ~ **Z** of Part III blank.

IV

Q 1 Let a be a constant. Assume that the function

$$f(x) = 2\sin^3 x + a\sin 2x + \frac{9}{2}\cos 2x - 9\cos x - 2ax + 6$$

takes a local extremum at $x = \frac{\pi}{3}$. We consider about the maximum and minimum values of $f(x)$ over the interval $0 \leqq x \leqq \frac{\pi}{2}$.

(1) Since $f(x)$ takes a local extremum at $x = \frac{\pi}{3}$, it follows that $a = \frac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}$.

Hence the derivative $f'(x)$ of $f(x)$ can be expressed as

$$f'(x) = \boxed{\mathsf{C}}\sin x\left(\boxed{\mathsf{D}}\cos x - 1\right)\left(\sin x - \boxed{\mathsf{E}}\right).$$

(2) It can be seen from the result of (1) that $f(x)$ over $0 \leqq x \leqq \frac{\pi}{2}$ takes the maximum value at $x = \boxed{\mathsf{F}}$ and the minimum value at $x = \boxed{\mathsf{G}}$, where $\boxed{\mathsf{F}}$ and $\boxed{\mathsf{G}}$ are the appropriate expressions from among ⓪ ~ ④ below.

⓪ 0 ① $\frac{\pi}{6}$ ② $\frac{\pi}{4}$ ③ $\frac{\pi}{3}$ ④ $\frac{\pi}{2}$

- memo -

Q 2 A sequence $\{a_n\}$ is defined as

$$a_n = \int_0^{\frac{1}{4}} x^n e^{-x} dx \qquad (n = 1, 2, 3, \cdots).$$

Then

$$a_1 = -\frac{\boxed{\mathsf{H}}}{\boxed{\mathsf{I}}} e^{\frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}} + 1.$$

Also a_{n+1} can be expressed in terms of a_n as

$$a_{n+1} = -\left(\frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}}\right)^{n+1} e^{\frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}} + \left(n + \boxed{\mathsf{O}}\right) a_n \qquad (n = 1, 2, 3, \cdots).$$

When this is transformed into

$$n a_n = a_{n+1} - a_n + \left(\frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}}\right)^{n+1} e^{\frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}},$$

we have

$$\sum_{k=1}^{n} k a_k = a_{n+1} - a_1 + \frac{\boxed{\mathsf{P}}}{\boxed{\mathsf{QR}}} e^{\frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}} \left\{ 1 - \left(\frac{\boxed{\mathsf{S}}}{\boxed{\mathsf{T}}}\right)^n \right\}.$$

Since, for $0 \leqq x$, the range of values of e^{-x} is $0 < e^{-x} \leqq \boxed{\mathsf{U}}$, it follows that

$$0 < a_n < \int_0^{\frac{1}{4}} \boxed{\mathsf{U}}\, x^n \, dx = \frac{1}{\boxed{\mathsf{V}}^{\,n+1}(n+1)}.$$

Thus, since

$$\lim_{n \to \infty} a_n = \boxed{\mathsf{W}},$$

we obtain

$$\lim_{n \to \infty} \sum_{k=1}^{n} k a_k = \frac{\boxed{\mathsf{X}}}{\boxed{\mathsf{Y}}} e^{\frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}} - 1.$$

- memo -

This is the end of the questions for Part IV. Leave the answer space **Z** of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.