Question Booklet

2013 Examination for Japanese University Admission
for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter A, B, C, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$ is used later in the question. In such a case, the symbol is shaded when it is used later, as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$.

Note the following:

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
$-\frac{2}{\sqrt{6}}=\frac{-2\sqrt{6}}{6}=\frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{\text{A}}\sqrt{\boxed{\text{B}}}}{\boxed{\text{C}}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to $\boxed{\text{DE}}\,x$ is $-x$, mark "−" for D and "1" for E as shown below.

A	●	0	1	2	3	4	5	6	7	8	9
B	⊖	0	1	2	●	4	5	6	7	8	9
C	⊖	0	1	2	3	●	5	6	7	8	9
D	●	0	1	2	3	4	5	6	7	8	9
E	⊖	0	●	2	3	4	5	6	7	8	9

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

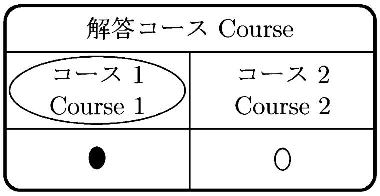
(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >



If you do not correctly fill in the appropriate oval, your answers will not be graded.

I

Q 1 Consider the quadratic function

$$y = -x^2 - ax + 3. \qquad \cdots\cdots\cdots \quad ①$$

(1) If $a > 0$ and the maximum value of function ① is 7, then $a =$ **A** . In this case, the equation of the axis of symmetry of the graph is $x =$ **BC** , and the x-coordinates of the points of intersection of this graph and the x-axis are **DE** $\pm\sqrt{\textbf{F}}$.

(2) If the curve obtained by translating the graph of function ① by 2 in the x-direction and by -3 in the y-direction passes through $(-3, -5)$, then $a =$ **G** .

- memo -

Q 2 For $\boxed{\mathbf{H}}$, $\boxed{\mathbf{I}}$ in question (1), and for $\boxed{\mathbf{J}}$, $\boxed{\mathbf{K}}$ in question (2), choose the appropriate answer from among ⓪ ～ ③ at the bottom of this page.

For $\boxed{\mathbf{L}}$ ～ $\boxed{\mathbf{R}}$ in question (3), enter the appropriate number.

Consider the following three possible conditions on two real numbers x and y:

p : x and y satisfy the equation $(x+y)^2 = a(x^2+y^2)+bxy$,
where a and b are real constants.

q : $x=0$ and $y=0$.

r : $x=0$ or $y=0$.

(1) Suppose that in condition p, $a=b=1$. Then p is $\boxed{\mathbf{H}}$ for q, and p is $\boxed{\mathbf{I}}$ for r.

(2) Suppose that in condition p, $a=b=2$. Then p is $\boxed{\mathbf{J}}$ for q, and p is $\boxed{\mathbf{K}}$ for r.

(3) If in condition p we set $a=2$, we can transform the equation in p into

$$\left(x + \frac{b-\boxed{\mathbf{L}}}{\boxed{\mathbf{M}}}\,y\right)^2 + \left(\boxed{\mathbf{N}} - \frac{\left(b-\boxed{\mathbf{O}}\right)^2}{\boxed{\mathbf{P}}}\right) y^2 = 0.$$

Hence p is a necessary and sufficient condition for q if and only if b satisfies

$$\boxed{\mathbf{Q}} < b < \boxed{\mathbf{R}}.$$

⓪ a necessary and sufficient condition

① a necessary condition but not a sufficient condition

② a sufficient condition but not a necessary condition

③ neither a necessary condition nor a sufficient condition

- memo -

This is the end of the questions for Part I. Leave the answer spaces **S** ～ **Z** of Part I blank.

II

Q 1 In a bag there are a total of nine balls: one white, three red and five black. The white ball is worth five points, a red ball is worth three points, and a black ball is worth one point. Two balls are taken from the bag together, and the trial is scored by the sum of the values of the two balls.

(1) The highest possible score is **A**, and the probability that it happens is $\dfrac{\boxed{\textbf{B}}}{\boxed{\textbf{CD}}}$.

(2) The probability that the score is 6 is $\dfrac{\boxed{\textbf{E}}}{\boxed{\textbf{F}}}$.

(3) The expected value of the score is $\dfrac{\boxed{\textbf{GH}}}{\boxed{\textbf{I}}}$.

- memo -

Q 2 A natural number n is said to be a perfect square when there exists a natural number x satisfying $n = x^2$. Similarly, n is said to be a perfect cube when there exists a natural number x satisfying $n = x^3$.

In the following two cases, find the natural number n that satisfies the conditions.

(i) n is a perfect square. Furthermore, the number obtained by adding 13 to n is also a perfect square.

(ii) n is a perfect cube. Furthermore, the number obtained by adding 61 to n is also a perfect cube.

First, consider (i). From the definition of a perfect square number, n can be expressed as $n = x^2$, where x is a natural number. In addition, there exists a natural number y such that

$$x^2 + 13 = y^2.$$

Since $x < y$, $y - x =$ **J** and $y + x =$ **KL**. It follows that

$$x = \boxed{\mathsf{M}}, \quad y = \boxed{\mathsf{N}},$$

and finally that $n =$ **OP**.

Next, consider (ii). Similar to (i), in (ii), there exists a natural number x such that $n = x^3$, and there also exists a natural number y such that

$$x^3 + 61 = y^3.$$

When we solve this equation, we obtain

$$x = \boxed{\mathsf{Q}}, \quad y = \boxed{\mathsf{R}},$$

and hence the perfect cube $n =$ **ST**.

- memo -

This is the end of the questions for Part [II]. Leave the answer spaces [U] ~ [Z] of Part [II] blank.

III

Let a be a constant. Consider the quadratic inequality

$$x^2 - 2(a+2)x + 25 > 0. \qquad \cdots\cdots\cdots \quad ①$$

The left-hand side of inequality ① can be transformed into

$$\left(x - a - \boxed{\mathsf{A}}\right)^2 - a^2 - \boxed{\mathsf{B}}\,a + \boxed{\mathsf{CD}}.$$

Hence, we have the following results.

(1) The condition under which inequality ① holds for all real numbers x is

$$\boxed{\mathsf{EF}} < a < \boxed{\mathsf{G}}.$$

(2) The condition under which inequality ① holds for all real numbers x satisfying $x \geqq -1$ is

$$\boxed{\mathsf{HIJ}} < a < \boxed{\mathsf{K}}.$$

- memo -

This is the end of the questions for Part III. Leave the answer spaces **L** ~ **Z** of Part III blank.

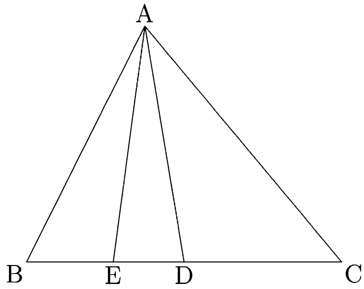
IV

Suppose that in the figure to the right

$$\mathrm{AB} = 4, \quad \mathrm{AC} = 5, \quad \cos\angle\mathrm{BAC} = \frac{1}{8}$$

and

$$\angle\mathrm{BAD} = \angle\mathrm{ACB}, \quad \angle\mathrm{CAE} = \angle\mathrm{ABC}.$$



(1) When we denote the area of $\triangle$ABC by S, we have

$$S = \frac{\boxed{\mathsf{AB}}\sqrt{\boxed{\mathsf{C}}}}{\boxed{\mathsf{D}}}.$$

Also BC = $\boxed{\mathsf{E}}$.

(2) Furthermore, when we denote the areas of $\triangle$ABD and $\triangle$ACE by S_1 and S_2, respectively, we have

$$S : S_1 : S_2 = 1 : \frac{\boxed{\mathsf{F}}}{\boxed{\mathsf{G}}} : \frac{\boxed{\mathsf{HI}}}{\boxed{\mathsf{JK}}}.$$

(3) When we denote the area of $\triangle$ADE by T, we have

$$T = \frac{\boxed{\mathsf{LM}}\sqrt{\boxed{\mathsf{N}}}}{\boxed{\mathsf{OP}}}.$$

Also $\mathrm{DE} = \dfrac{\boxed{\mathsf{Q}}}{\boxed{\mathsf{R}}}$.

- memo -

This is the end of the questions for Part IV.

Leave the answer spaces **S** ～ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

Do not take this question booklet out of the room.

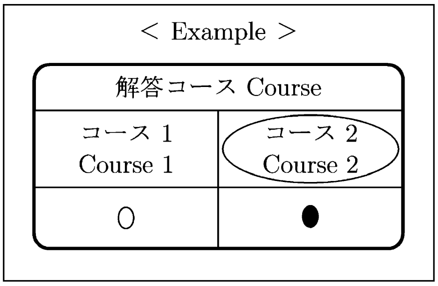
Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.



<u>**If you do not correctly fill in the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Consider the quadratic function

$$y = -x^2 - ax + 3. \qquad \cdots\cdots\cdots \quad ①$$

(1) If $a > 0$ and the maximum value of function ① is 7, then $a =$ **A**. In this case, the equation of the axis of symmetry of the graph is $x =$ **BC**, and the x-coordinates of the points of intersection of this graph and the x-axis are **DE** $\pm \sqrt{\textbf{F}}$.

(2) If the curve obtained by translating the graph of function ① by 2 in the x-direction and by -3 in the y-direction passes through $(-3, -5)$, then $a =$ **G**.

- memo -

Q 2 For **H**, **I** in question (1), and for **J**, **K** in question (2), choose the appropriate answer from among ⓪ ～ ③ at the bottom of this page.

For **L** ～ **R** in question (3), enter the appropriate number.

Consider the following three possible conditions on two real numbers x and y:

p : x and y satisfy the equation $(x+y)^2 = a(x^2+y^2)+bxy$,
where a and b are real constants.

q : $x=0$ and $y=0$.

r : $x=0$ or $y=0$.

(1) Suppose that in condition p, $a=b=1$. Then p is **H** for q, and p is **I** for r.

(2) Suppose that in condition p, $a=b=2$. Then p is **J** for q, and p is **K** for r.

(3) If in condition p we set $a=2$, we can transform the equation in p into

$$\left(x+\frac{b-\boxed{\mathsf{L}}}{\boxed{\mathsf{M}}}y\right)^2+\left(\boxed{\mathsf{N}}-\frac{\left(b-\boxed{\mathsf{O}}\right)^2}{\boxed{\mathsf{P}}}\right)y^2=0.$$

Hence p is a necessary and sufficient condition for q if and only if b satisfies

$$\boxed{\mathsf{Q}}<b<\boxed{\mathsf{R}}.$$

⓪ a necessary and sufficient condition

① a necessary condition but not a sufficient condition

② a sufficient condition but not a necessary condition

③ neither a necessary condition nor a sufficient condition

- memo -

This is the end of the questions for Part I. Leave the answer spaces **S** ~ **Z** of Part I blank.

II

Let the sequence $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ be an arithmetic progression satisfying

$$a_2 = 2, \quad a_6 = 3a_3.$$

Then, consider the series $\sum_{n=1}^{\infty} \frac{3^n}{r^{a_n}}$, where r is a positive real number.

(1) When we denote the first term of $\{a_n\}$ by a, and the common difference by d, we have

$$a = \boxed{\textsf{AB}}, \quad d = \boxed{\textsf{C}}.$$

(2) The series $\sum_{n=1}^{\infty} \frac{3^n}{r^{a_n}}$ is an infinite geometric series where the first term is $\boxed{\textsf{D}}\, r^{\boxed{\textsf{E}}}$,

and the common ratio is $\dfrac{\boxed{\textsf{F}}}{r^{\boxed{\textsf{G}}}}$. Hence, this series converges when

$$r > 3^{\frac{\boxed{\textsf{H}}}{\boxed{\textsf{I}}}},$$

and its sum S is

$$S = \frac{\boxed{\textsf{J}}\, r^{\boxed{\textsf{K}}}}{r^{\boxed{\textsf{L}}} - \boxed{\textsf{M}}}.$$

(3) This sum S is minimized at

$$r = \boxed{\textsf{N}}^{\frac{\boxed{\textsf{O}}}{2}}.$$

- memo -

This is the end of the questions for Part II. Leave the answer spaces **P** ~ **Z** of Part II blank.

III

Consider the function

$$f(x) = \sin 2x - 3(\sin x + \cos x)$$

on the interval $-\dfrac{\pi}{3} \leqq x \leqq \dfrac{\pi}{3}$.

(1) Let $t = \sin x + \cos x$. The range of the values which t can take is

$$\frac{\boxed{\mathsf{A}} - \sqrt{\boxed{\mathsf{B}}}}{\boxed{\mathsf{C}}} \leqq t \leqq \sqrt{\boxed{\mathsf{D}}}.$$

(2) The function $f(x)$ takes its minimum value $\boxed{\mathsf{E}} - \boxed{\mathsf{F}}\sqrt{\boxed{\mathsf{G}}}$ at $x = \dfrac{\boxed{\mathsf{H}}}{\boxed{\mathsf{I}}}\pi$.

- memo -

This is the end of the questions for Part III. Leave the answer spaces **J** ～ **Z** of Part III blank.

IV

Q 1 For each of **A** ~ **I** in the following sentences, choose the appropriate answer from among ⓪ ~ ⑨ at the bottom of this page.

We are to compare the magnitudes of a^{a+1} and $(a+1)^{a}$ by using the properties of the function $f(x) = \dfrac{\log x}{x}$, where $a > 0$.

(1) Since the derivative of $f(x)$ is

$$f'(x) = \frac{\boxed{\text{A}} - \log x}{x^{\boxed{\text{B}}}},$$

the interval on x in which $f(x)$ monotonically increases is

$$\boxed{\text{C}} < x \leqq \boxed{\text{D}},$$

and the interval on x in which $f(x)$ monotonically decreases is

$$\boxed{\text{E}} \leqq x.$$

(2) When we set $p = a^{a+1}$, $q = (a+1)^{a}$, we have

$$\log p - \log q = \left(a^{\boxed{\text{F}}} + a\right)\left\{f(a) - f\left(a + \boxed{\text{G}}\right)\right\}.$$

Hence we see that

$$\text{if} \quad 0 < a < \frac{3}{2} \quad \text{then} \quad p \ \boxed{\text{H}} \ q,$$

and

$$\text{if} \quad 3 < a \quad \text{then} \quad p \ \boxed{\text{I}} \ q.$$

⓪	0	①	1	②	2	③	3
④	e	⑤	$e+1$	⑥	$\dfrac{1}{e}$		
⑦	$>$	⑧	$=$	⑨	$<$		

- memo -

Q 2 Let $0 < a < 1$. Let $S(a)$ denote the sum of the areas of two regions, one region bounded by the curve $y = xe^{2x}$, the x-axis, and the straight line $x = a - 1$, and the other region bounded by the curve $y = xe^{2x}$, the x-axis, and the straight line $x = a$. We are to find the value of a at which $S(a)$ is minimized.

The indefinite integral of xe^{2x} is

$$\frac{\boxed{\mathsf{J}}}{\boxed{\mathsf{K}}}\left(\boxed{\mathsf{L}}\,x - 1\right)e^{2x} + C,$$

where C is the constant of integration.

The value of xe^{2x} is $xe^{2x} < 0$ for $x < 0$ and $xe^{2x} \geqq 0$ for $x \geqq 0$. Hence we have

$$S(a) = \frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}}\left\{\boxed{\mathsf{O}} + \left(\boxed{\mathsf{P}}\,a - \boxed{\mathsf{Q}}\right)e^{2(a-1)} + \left(\boxed{\mathsf{R}}\,a - 1\right)e^{2a}\right\}.$$

Further, since

$$S'(a) = \left(a - \boxed{\mathsf{S}}\right)e^{2(a-1)} + ae^{2a},$$

the value of a at which $S(a)$ is minimized is $a = \dfrac{\boxed{\mathsf{T}}}{e^2 + \boxed{\mathsf{U}}}$, which satisfies $0 < a < 1$.

- memo -

This is the end of the questions for Part IV.

Leave the answer spaces **V** ~ **Z** of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.