Question Booklet

2014 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter A, B, C, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such a case, the symbol is shaded when it is used later, as [A] or [BC].

Note the following:

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to [DE] x is $-x$, mark "−" for D and "1" for E as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

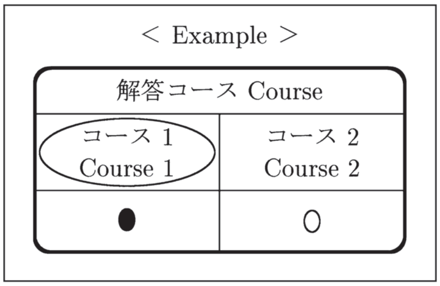
(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.



If you do not correctly fill in the appropriate oval, your answers will not be graded.

I

Q 1 A quadratic function $y = ax^2 + bx + \frac{3}{a}$ satisfies the following two conditions:

(i) y is maximized at $x = 3$,

(ii) the value of y at $x = 1$ is 2.

We are to find the values of a and b.

Using conditions (i) and (ii), we obtain the following relationships between a and b:

$$\begin{cases} b = \boxed{\mathsf{AB}}\,a \\ \boxed{\mathsf{C}} = a + b + \dfrac{\boxed{\mathsf{D}}}{a}. \end{cases}$$

From these two equalities, we have the equation

$$\boxed{\mathsf{E}}\,a^2 + \boxed{\mathsf{F}}\,a - \boxed{\mathsf{G}} = 0$$

and hence

$$a = \boxed{\mathsf{HI}}, \quad b = \boxed{\mathsf{J}}.$$

Thus the maximum value of this function is $\boxed{\mathsf{K}}$.

- memo -

Q 2 Consider

$$E = P^2 - 4Q^2 - 3P + 6Q$$

where P and Q are the integral expressions

$$P = 2x^2 - x + 2, \quad Q = x^2 - 2x + 1.$$

(1) By factorizing the right side of E, we obtain

$$E = (P - \boxed{\textbf{L}}\,Q)(P + \boxed{\textbf{M}}\,Q - \boxed{\textbf{N}}).$$

(2) When we express E in terms of x, we have

$$E = \boxed{\textbf{O}}\,x(x - \boxed{\textbf{P}})(\boxed{\textbf{Q}}\,x - \boxed{\textbf{R}}).$$

(3) If $x = -\dfrac{1-\sqrt{5}}{3-\sqrt{5}}$, then the value of E is $\boxed{\textbf{S}} + \boxed{\textbf{T}}\sqrt{\boxed{\textbf{U}}}$.

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This is the end of the questions for Part I. Leave the answer spaces **V** ～ **Z** of Part I blank.

II

Q 1 In a box, there are n red balls and $(20 - n)$ white balls, where $0 < n < 20$. In each trial, a ball is taken out of the box, its color is examined, and it is returned to the box.

(1) Let x be the probability that the ball taken out in one trial is red. Then, $x = \dfrac{n}{\boxed{\mathsf{AB}}}$.

(2) Let p be the probability that in two trials a white ball is taken out at least once. Then p can be expressed as $p = \boxed{\mathsf{C}} - x^{\boxed{\mathsf{D}}}$, where x is the x of (1).

(3) Let q be the probability that in four trials a white ball is taken out at least twice. Then q can be expressed as

$$q = \boxed{\mathsf{E}} - \boxed{\mathsf{F}}\,x^{\boxed{\mathsf{G}}} + \boxed{\mathsf{H}}\,x^{\boxed{\mathsf{I}}},$$

where x is the x of (1).

(4) For p and q of (2) and (3), we are to find the maximum value of n such that $p < q$.

From the inequality $p < q$, we obtain the inequality

$$\boxed{\mathsf{J}}\,x^2 - \boxed{\mathsf{K}}\,x + 1 > 0.$$

When we solve this, we have

$$x < \frac{1}{\boxed{\mathsf{L}}}.$$

Thus the maximum value of n is $\boxed{\mathsf{M}}$.

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Q 2 Let p be a prime number, and let x and y be positive integers. Then we are to find all triples of p, x and y which satisfy

$$\frac{p}{x} + \frac{7}{y} = p.$$

We can transform this equation into

$$(x - \boxed{\mathbf{N}})(py - \boxed{\mathbf{O}}) = \boxed{\mathbf{P}}.$$

From this, we obtain

$$x - \boxed{\text{N}} = \boxed{\mathbf{Q}} \text{ or } \boxed{\mathbf{R}}, \qquad \text{(note: have } \boxed{\text{Q}} < \boxed{\text{R}}\text{)}$$

and hence

$$x = \boxed{\mathbf{S}} \text{ or } \boxed{\mathbf{T}}. \qquad \text{(note: have } \boxed{\text{S}} < \boxed{\text{T}}\text{)}$$

First, if $x = \boxed{\text{S}}$, then

$$p = \boxed{\mathbf{U}}, \quad y = \boxed{\mathbf{V}}$$

or

$$p = \boxed{\mathbf{W}}, \quad y = \boxed{\mathbf{X}}. \qquad \text{(note: have } \boxed{\text{U}} < \boxed{\text{W}}\text{)}$$

Next, if $x = \boxed{\text{T}}$, then

$$p = \boxed{\mathbf{Y}}, \quad y = \boxed{\mathbf{Z}}.$$

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This is the end of the questions for Part [II].

III

Consider a quadratic function in x

$$y = ax^2 + bx + c \qquad \cdots\cdots\cdots \quad ①$$

such that the graph of function ① passes through the two points $(-1, -1)$ and $(2, 2)$.

(1) When we express b and c in terms of a, we have

$$b = \boxed{\mathsf{A}} - a\,, \quad c = \boxed{\mathsf{BC}}\,a.$$

(2) Suppose that one of the points of intersection of the graph of function ① and the x-axis is within the interval $0 < x \leqq 1$. Then the range of values of a is

$$\boxed{\mathsf{D}} < a \leqq \frac{\boxed{\mathsf{E}}}{\boxed{\mathsf{F}}}. \qquad \cdots\cdots\cdots \quad ②$$

(3) When the value of a varies within interval ②, the range of values of $a + bc$ is

$$\frac{\boxed{\mathsf{GH}}}{\boxed{\mathsf{I}}} \leqq a + bc \leqq \boxed{\mathsf{J}}.$$

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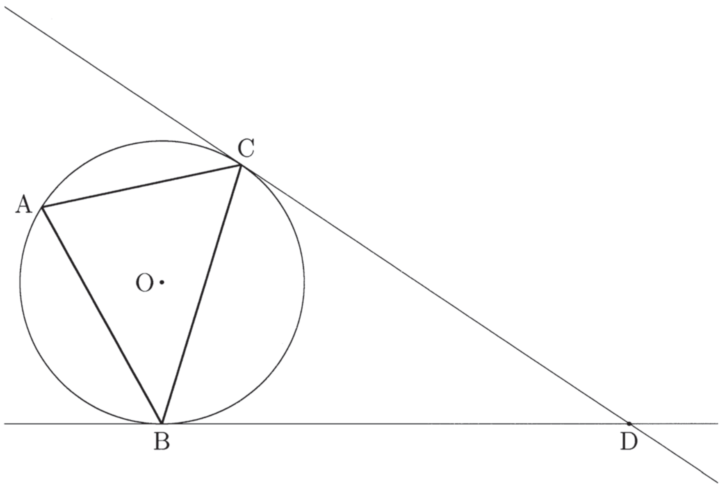
This is the end of the questions for Part III. Leave the answer spaces **K** ～ **Z** of Part III blank.

IV

A triangle ABC satisfies

$$AB = 7, \quad BC = 8, \quad CA = 6.$$

We denote the center and the radius of the circumscribed circle of this triangle ABC by O and r, respectively. We draw two straight lines which are tangent to this circumscribed circle at the points B and C, and denote the point of intersection of these straight lines by D.



We see that

$$\cos \angle BAC = \frac{\boxed{\mathbf{A}}}{\boxed{\mathbf{B}}}, \qquad \sin \angle BAC = \frac{\sqrt{\boxed{\mathbf{CD}}}}{\boxed{\mathbf{E}}},$$

$$r = \frac{\boxed{\mathbf{FG}}\sqrt{\boxed{\mathbf{HI}}}}{\boxed{\mathbf{JK}}}, \qquad BD = \boxed{\mathbf{LM}}.$$

Furthermore, if P is a point on the circumscribed circle, the shortest possible length of the segment DP is $\dfrac{\boxed{\mathbf{NO}}\sqrt{\boxed{\mathbf{PQ}}}}{\boxed{\mathbf{R}}}$.

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This is the end of the questions for Part IV.

Leave the answer spaces **S** ~ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

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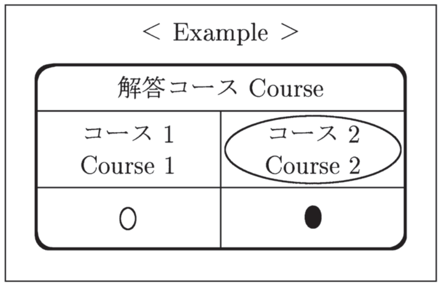

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and hence

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$$E = \boxed{\mathsf{O}}\,x\,(x - \boxed{\mathsf{P}})(\boxed{\mathsf{Q}}\,x - \boxed{\mathsf{R}}).$$

(3) If $x = -\dfrac{1-\sqrt{5}}{3-\sqrt{5}}$, then the value of E is $\boxed{\mathsf{S}} + \boxed{\mathsf{T}}\sqrt{\boxed{\mathsf{U}}}$.

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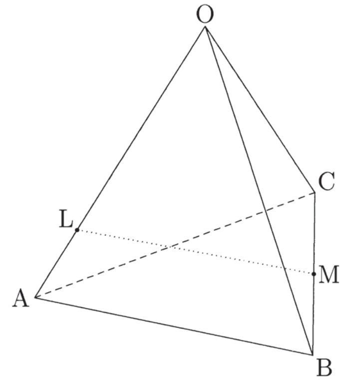
This is the end of the questions for Part I. Leave the answer spaces **V** ～ **Z** of Part I blank.

II

In a regular tetrahedron OABC, each side of which has the length 1, let L denote the point which divides segment OA internally in the ratio 3 : 1, let M denote the midpoint of side BC, and let P denote the point which divides segment LM internally in the ratio $t : (1-t)$, where $0 < t < 1$.



(1) When we set $\overrightarrow{\mathrm{OA}} = \vec{a}$, $\overrightarrow{\mathrm{OB}} = \vec{b}$, $\overrightarrow{\mathrm{OC}} = \vec{c}$ and express $\overrightarrow{\mathrm{OP}}$ in terms of $\vec{a}$, $\vec{b}$, $\vec{c}$, we have

$$\overrightarrow{\mathrm{OP}} = \frac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}\left(\boxed{\mathsf{C}} - t\right)\vec{a} + \frac{\boxed{\mathsf{D}}}{\boxed{\mathsf{E}}}t\left(\vec{b} + \vec{c}\right).$$

Since, $\vec{a}\cdot\left(\vec{b}+\vec{c}\right) = \boxed{\mathsf{F}}$ and $\left|\vec{b}+\vec{c}\right|^2 = \boxed{\mathsf{G}}$, we have

$$\left|\overrightarrow{\mathrm{OP}}\right| = \frac{1}{\boxed{\mathsf{H}}}\sqrt{\boxed{\mathsf{I}}\,t^2 - \boxed{\mathsf{J}}\,t + \boxed{\mathsf{K}}},$$

where $\vec{a}\cdot\left(\vec{b}+\vec{c}\right)$ means the inner product of the vectors $\vec{a}$ and $\left(\vec{b}+\vec{c}\right)$.

(2) The value of t at which $\left|\overrightarrow{\mathrm{OP}}\right|$ is minimized is

$$t = \frac{\boxed{\mathsf{L}}}{\boxed{\mathsf{M}}},$$

and the minimum value of $\left|\overrightarrow{\mathrm{OP}}\right|$ is $\dfrac{\sqrt{\boxed{\mathsf{N}}}}{\boxed{\mathsf{O}}}$.

(3) When $\left|\overrightarrow{\mathrm{OP}}\right|$ is minimized as in (2), we have $\cos\angle\mathrm{AOP} = \dfrac{\boxed{\mathsf{P}}\sqrt{\boxed{\mathsf{Q}}}}{\boxed{\mathsf{R}}}$.

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This is the end of the questions for Part II. Leave the answer spaces **S** ～ **Z** of Part II blank.

III

Consider the following two equations in x

$$\sin 2x + a\cos x = 0 \quad \cdots\cdots\cdots \quad ①$$

$$\cos 2x + a\sin x = -2 \quad \cdots\cdots\cdots \quad ②$$

over the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, where $a > 0$.

Let $a = \sqrt{2}$. Then the value of x which satisfies ① is

$$x = \frac{\boxed{\text{AB}}}{\boxed{\text{C}}}\pi.$$

However, at this x the value of the left side of ② is $\boxed{\text{DE}}$, and so equation ② does not hold. Hence, when $a = \sqrt{2}$, ① and ② have no common solution.

Now, let us find a value of a such that ① and ② have a common solution, and also the common solution x.

First, from ① we have

$$\sin x = \frac{\boxed{\text{FG}}}{\boxed{\text{H}}}a, \quad \cos 2x = \boxed{\text{I}} - \frac{a^2}{\boxed{\text{J}}}.$$

When we substitute these into ②, we obtain

$$a^2 = \boxed{\text{K}}.$$

Thus $a = \sqrt{\boxed{\text{K}}}$, and the common solution is

$$x = \frac{\boxed{\text{LM}}}{\boxed{\text{N}}}\pi.$$

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This is the end of the questions for Part III. Leave the answer spaces O ～ Z of Part III blank.

IV

Q 1 Let $a > 0$. Consider two curves

$$C_1:\quad y = e^{6x}$$
$$C_2:\quad y = ax^2.$$

We are to find the condition on a such that there exist two straight lines, each of which is tangent to both C_1 and C_2.

The equation of the tangent to C_1 at a point (t, e^{6t}) is

$$y = \boxed{\mathbf{A}}\, e^{6t}x - e^{6t}\left(\boxed{\mathbf{B}}\, t - \boxed{\mathbf{C}}\right).$$

This is tangent also to C_2 under the condition that the quadratic equation

$$ax^2 = \boxed{\text{A}}\, e^{6t}x - e^{6t}\left(\boxed{\text{B}}\, t - \boxed{\text{C}}\right)$$

has just one solution. Hence, the equation

$$\boxed{\mathbf{D}}\, e^{12t} - ae^{6t}\left(\boxed{\mathbf{E}}\, t - \boxed{\mathbf{F}}\right) = 0$$

must hold for a and t. From this equation we obtain

$$a = \frac{\boxed{\text{D}}\, e^{6t}}{\boxed{\text{E}}\, t - \boxed{\text{F}}}.$$

Let $f(t)$ denote the right side of this equation. The condition under which there exist two straight lines each of which is tangent to both C_1 and C_2, is that the straight line $s = a$ intersects the graph of $s = f(t)$ at two points.

Now, the derivative of $f(t)$ is

$$f'(t) = \frac{108\,e^{6t}\left(\boxed{\mathbf{G}}\, t - \boxed{\mathbf{H}}\right)}{\left(\boxed{\text{E}}\, t - \boxed{\text{F}}\right)^2}.$$

Hence the condition on a that we are seeking is

$$a > \boxed{\mathbf{I}}\, e^{\boxed{\mathbf{J}}}.$$

Note that $\displaystyle\lim_{t\to\infty} \frac{e^t}{t} = \infty$.

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Q 2 For **K** ~ **Z** in the following statements, choose the appropriate answer from among ⓪ ~ ⑨ at the bottom of this page.

Let a and t be positive real numbers. Let D denote the region of a plane bounded by the graph of the quadratic function in x

$$y = \frac{1}{t^2}\left(x - at^2\right)^2,$$

the x-axis, and the y-axis. Let V_1 denote the volume of the solid obtained by rotating D once about the x-axis, and V_2 denote the volume of the solid obtained by rotating D once about the y-axis. Now, let us show that for a certain value of a, $V_1 = V_2$, independent of the value of t.

First, the value of V_1 is

$$V_1 = \pi \int_{\boxed{K}}^{\boxed{L}} \frac{1}{t^{\boxed{M}}}\left(x - at^2\right)^{\boxed{N}} dx$$

$$= \frac{\pi}{\boxed{O}}\, a^{\boxed{P}}\, t^{\boxed{Q}}.$$

Next, the value of V_2 is

$$V_2 = \pi \int_{\boxed{K}}^{\boxed{R}} \left(\boxed{S} - \boxed{T}\sqrt{y}\right)^{\boxed{U}} dy$$

$$= \frac{\pi}{\boxed{V}}\, a^{\boxed{W}}\, t^{\boxed{X}}.$$

Hence, when $a = \dfrac{\boxed{Y}}{\boxed{Z}}$, then $V_1 = V_2$, independent of the value of t.

⓪ 0	① 1	② 2	③ 3	④ 4
⑤ 5	⑥ 6	⑦ t	⑧ at^2	⑨ a^2t^2

- memo -

This is the end of the questions for Part IV.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.