Question Booklet

2011 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instruction for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instruction for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). Completely fill in your answer for each letter in the corresponding line of the answer sheet (mark-sheet).

Note the following :

(1) Express square roots ($\sqrt{\ }$) in their simplest form.
(Example : Substitute $2\sqrt{3}$ for $\sqrt{12}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to $\boxed{DE}\,x$ is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

3. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

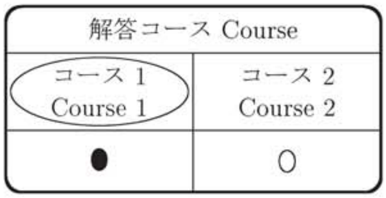
(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label "Course 1" and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >



<u>**If you do not correctly black out the appropriate oval, your answers will not be graded.**</u>

I

Q 1 For the real numbers a and b satisfying

$$a^3 = \frac{1}{\sqrt{5}-2}, \quad b^3 = 2-\sqrt{5},$$

we are to find the value of $a+b$.

When we set $x = a+b$, we have

$$x^3 = (a+b)^3 = a^3 + b^3 + \boxed{\text{A}}\,ab(a+b).$$

Since $ab = \boxed{\text{BC}}$, we know that this x satisfies

$$x^3 + \boxed{\text{D}}\,x - \boxed{\text{E}} = 0.$$

The left side of this equation can be factorized as follows:

$$\begin{aligned} x^3 + \boxed{\text{D}}\,x - \boxed{\text{E}} &= \left(x^3 - \boxed{\text{F}}\right) + \boxed{\text{D}}\left(x - \boxed{\text{F}}\right) \\ &= \left(x - \boxed{\text{F}}\right)\left(x^2 + x + \boxed{\text{G}}\right). \end{aligned}$$

Since

$$x^2 + x + \boxed{\text{G}} = \left(x + \frac{\boxed{\text{H}}}{\boxed{\text{I}}}\right)^2 + \frac{\boxed{\text{JK}}}{\boxed{\text{L}}} > 0,$$

we obtain $x = a+b = \boxed{\text{M}}$.

- memo -

Q 2 Consider the two functions $y = x^2 + ax + a$ and $y = x + 1$.

(1) The number of points at which the graphs of the two functions meet depends on the relationship of a with the numbers **Q** and **R** in the following way: (For **N** ～ **P** choose which of ⓪ ～ ② gives the correct condition for the question.)

(i) The condition under which the graphs of the two functions intersect at two different points is **N**.

(ii) The condition under which the graphs of the two functions are tangent at a point is **O**.

(iii) The condition under which the graph of $y = x^2 + ax + a$ is always above the graph of $y = x + 1$ is **P**.

⓪ **Q** $< a <$ **R**

① $a =$ Q or $a =$ R

② $a <$ Q or R $< a$

(2) Let us consider the case where the value of a satisfies P. Let $g(x)$ be the difference between the values of the two functions, so $g(x) = x^2 + ax + a - (x + 1)$, and let m be the minimum value of $g(x)$. Then

$$m = -\frac{\boxed{\mathsf{S}}}{\boxed{\mathsf{T}}}\left(a^2 - \boxed{\mathsf{U}}\,a + \boxed{\mathsf{V}}\right).$$

Hence m takes the maximum at $a =$ **W** and its value there is $m =$ **X**.

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This is the end of the questions for Part I. Leave the answer spaces Y, Z of Part I blank.

II

Q 1 There are six boxes numbered from 1 to 6. We are to put four balls of different sizes into these boxes.

(1) There are altogether $\boxed{\mathsf{A}}^{\boxed{\mathsf{B}}}$ ways to put the four balls into the boxes.

(2) There are $\boxed{\mathsf{CDE}}$ ways to put the four balls into four separate boxes.

(3) There are $\boxed{\mathsf{FGH}}$ ways to put three balls into one box and the fourth ball into another.

(4) There are $\boxed{\mathsf{IJK}}$ ways to put at least one ball into the box numbered 1.

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Q 2 Consider the quadratic function of x

$$x^2 + (4a - 6)x + 2a + b + 5 = 0. \qquad \cdots\cdots\cdots \quad ①$$

We are to find the conditions under which one solution is -1 and the other solution satisfies the inequality

$$|x + 2a| < a + 1. \qquad \cdots\cdots\cdots \quad ②$$

(1) The condition for equation ① to have the solution -1 is

$$b = \boxed{\mathsf{L}}\,a - \boxed{\mathsf{MN}}. \qquad \cdots\cdots\cdots \quad ③$$

Denote the other solution by α. When α is expressed in terms of a, we have

$$\alpha = \boxed{\mathsf{OP}}\,a + \boxed{\mathsf{Q}}.$$

(2) When $a > \boxed{\mathsf{RS}}$, the inequality ② has a solution, and its solution is

$$\boxed{\mathsf{TU}}\,a - \boxed{\mathsf{V}} < x < -a + \boxed{\mathsf{W}}.$$

Hence the conditions are: that a and b satisfy ③ and that a satisfies

$$\boxed{\mathsf{X}} < a < \boxed{\mathsf{Y}}.$$

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This is the end of the questions for Part II. Leave the answer space **Z** of Part II blank.

III

Consider two quadratic functions

$$y = 2x^2 + 3ax + 4b \qquad \cdots\cdots\cdots \quad ①$$

$$y = bx^2 + cx + d \qquad \cdots\cdots\cdots \quad ②$$

whose graphs are mutually symmetric with respect to the origin.

(1) From the symmetry with respect to the origin we see that

$$b = \boxed{\mathsf{AB}}, \quad c = \boxed{\mathsf{C}}\,a, \quad d = \boxed{\mathsf{D}}.$$

Hence ② can be reduced to

$$y = \boxed{\mathsf{AB}}\,x^2 + \boxed{\mathsf{C}}\,ax + \boxed{\mathsf{D}}. \qquad \cdots\cdots\cdots \quad ③$$

(2) Let $0 < a < 1$, and consider the graph of ③.

When the range of values of x is $0 \leqq x \leqq \frac{3}{2}$, the range of values of y in ③ is

$$\frac{\boxed{\mathsf{E}}}{\boxed{\mathsf{F}}}\,a + \frac{\boxed{\mathsf{G}}}{\boxed{\mathsf{H}}} \leqq y \leqq \frac{\boxed{\mathsf{I}}}{\boxed{\mathsf{J}}}\,a^2 + \boxed{\mathsf{K}}.$$

(3) For any value of a, the vertex of the graph of ③ is on the graph of the quadratic function

$$y = \boxed{\mathsf{L}}\,x^2 + \boxed{\mathsf{M}}.$$

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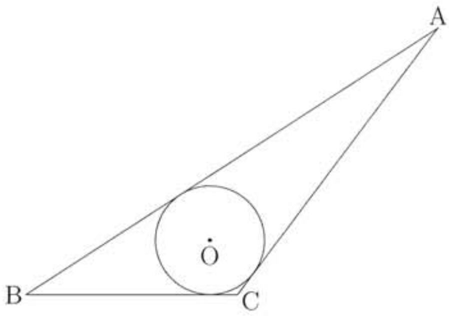
This is the end of the questions for Part III. Leave the answer spaces N ~ Z of Part III blank.

IV

Consider a triangle ABC where

$$\mathrm{AB} = 10, \quad \angle \mathrm{B} = 30^\circ$$

and the radius of its inscribed circle O is 1.



(1) Set $a = \mathrm{BC}$ and $b = \mathrm{CA}$. Finding the area S of the triangle ABC by two different methods, we have

$$S = \frac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}\, a,$$

and

$$S = \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}}\left(a + b + \boxed{\mathsf{EF}}\right).$$

Hence we obtain

$$b = \boxed{\mathsf{G}}\, a - \boxed{\mathsf{HI}}.$$

Since the relationship between a and b can also be expressed by the equation

$$b^2 = a^2 - \boxed{\mathsf{JK}}\sqrt{\boxed{\mathsf{L}}}\, a + \boxed{\mathsf{MNO}},$$

we have

$$a = \frac{\boxed{\mathsf{PQ}} - \boxed{\mathsf{R}}\sqrt{\boxed{\mathsf{S}}}}{3}, \quad b = \frac{\boxed{\mathsf{TU}} - \boxed{\mathsf{V}}\sqrt{\boxed{\mathsf{W}}}}{3}.$$

(2) Let D denote the point of intersection of the segment BC and the straight line which passes through the two points A and O. We denote the area of the triangle OBC by S'. Since

$$S : S' = \boxed{\mathsf{X}} : 1,$$

it follows that

$$\mathrm{AO} : \mathrm{OD} = \boxed{\mathsf{Y}} : 1.$$

- memo -

This is the end of the questions for Part IV. Leave the answer space **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

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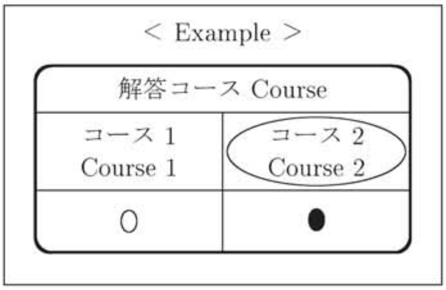

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The left side of this equation can be factorized as follows:

$$\begin{aligned} x^3 + \boxed{\text{D}}\, x - \boxed{\text{E}} &= \left(x^3 - \boxed{\text{F}}\right) + \boxed{\text{D}}\left(x - \boxed{\text{F}}\right) \\ &= \left(x - \boxed{\text{F}}\right)\left(x^2 + x + \boxed{\text{G}}\right). \end{aligned}$$

Since

$$x^2 + x + \boxed{\text{G}} = \left(x + \frac{\boxed{\text{H}}}{\boxed{\text{I}}}\right)^2 + \frac{\boxed{\text{JK}}}{\boxed{\text{L}}} > 0,$$

we obtain $x = a + b = \boxed{\text{M}}$.

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Q 2 Consider the two functions $y = x^2 + ax + a$ and $y = x + 1$.

(1) The number of points at which the graphs of the two functions meet depends on the relationship of a with the numbers **Q** and **R** in the following way: (For **N** ~ **P** choose which of ⓪ ~ ② gives the correct condition for the question.)

(i) The condition under which the graphs of the two functions intersect at two different points is **N**.

(ii) The condition under which the graphs of the two functions are tangent at a point is **O**.

(iii) The condition under which the graph of $y = x^2 + ax + a$ is always above the graph of $y = x + 1$ is **P**.

⓪ **Q** $< a <$ **R**

① $a =$ Q or $a =$ R

② $a <$ Q or R $< a$

(2) Let us consider the case where the value of a satisfies P. Let $g(x)$ be the difference between the values of the two functions, so $g(x) = x^2 + ax + a - (x + 1)$, and let m be the minimum value of $g(x)$. Then

$$m = -\frac{\boxed{\text{S}}}{\boxed{\text{T}}}\left(a^2 - \boxed{\text{U}}\,a + \boxed{\text{V}}\right).$$

Hence m takes the maximum at $a =$ **W** and its value there is $m =$ **X**.

- memo -

This is the end of the questions for Part I. Leave the answer spaces Y, Z of Part I blank.

II

Take four points

$$A(1,0),\quad B(0,1),\quad C(3,0),\quad D(0,2)$$

on a plane with the coordinate system having the origin O. Take two points P and Q on segments AB and CD respectively, such that

$$AP : PB = CQ : QD = k : 2.$$

We are to find the minimum possible length of the segment PQ.

(1) First, let us find the value of $x + 2y$, where $\overrightarrow{PQ} = (x, y)$.

Since

$$\overrightarrow{OP} = \frac{\boxed{\mathsf{A}}\,\overrightarrow{OA} + k\,\overrightarrow{OB}}{k + \boxed{\mathsf{B}}},\quad \overrightarrow{OQ} = \frac{\boxed{\mathsf{C}}\,\overrightarrow{OC} + k\,\overrightarrow{OD}}{k + \boxed{\mathsf{D}}},$$

we have

$$(x, y) = \frac{1}{k + \boxed{\mathsf{E}}}\left(\boxed{\mathsf{F}},\ k\right),$$

which gives $x + 2y = \boxed{\mathsf{G}}$.

(2) When we represent PQ^2 in terms of y, we have

$$PQ^2 = \boxed{\mathsf{H}}\,y^2 - \boxed{\mathsf{I}}\,y + \boxed{\mathsf{J}}.$$

Hence PQ takes the minimum at $y = \dfrac{\boxed{\mathsf{K}}}{\boxed{\mathsf{L}}}$ and its value there is

$PQ = \dfrac{\boxed{\mathsf{M}}\sqrt{\boxed{\mathsf{N}}}}{\boxed{\mathsf{O}}}$. In this case, the value of k is $k = \boxed{\mathsf{P}}$.

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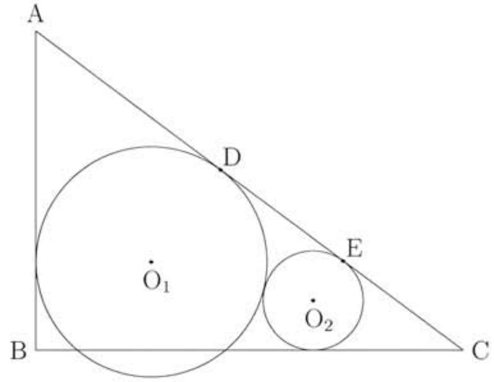
This is the end of the questions for Part II. Leave the answer spaces Q ~ Z of Part II blank.

III

Please look at the figure on the right.
We have a triangle ABC such that

$$AB = 9, \quad BC = 12, \quad \angle ABC = 90^\circ.$$

There are also two circles O_1 and O_2 with radii of length $2r$ and r, respectively. The two circles O_1 and O_2 are tangential to each other. Further, O_1 is tangent to the two sides AB and AC, and O_2 is tangent to the two sides CA and CB. We are to find the value of r.



First, let D and E denote the points at which the segment AC is tangent to the circles O_1 and O_2 respectively, and set $\alpha = \angle O_1AC$. Then, since $\tan 2\alpha = \dfrac{\boxed{\text{A}}}{\boxed{\text{B}}}$, we have $\tan\alpha = \dfrac{\boxed{\text{C}}}{\boxed{\text{D}}}$ using the double-angle formula. Thus, we obtain $AD = \boxed{\text{E}}\, r$.

Next, set $\beta = \angle O_2CA$. Since $\alpha + \beta = \boxed{\text{FG}}^\circ$, we have $\tan\beta = \dfrac{\boxed{\text{H}}}{\boxed{\text{I}}}$ using the addition theorem. Thus, we obtain $CE = \boxed{\text{J}}\, r$.

Moreover, it follows that $AC = \boxed{\text{KL}}$ and $DE = \boxed{\text{M}}\sqrt{\boxed{\text{N}}}\, r$.

Finally we obtain

$$r = \frac{\boxed{\text{OP}}\left(\boxed{\text{Q}} - \boxed{\text{R}}\sqrt{\boxed{\text{S}}}\right)}{41}.$$

- memo -

This is the end of the questions for Part III. Leave the answer spaces T ～ Z of Part III blank.

IV

Q 1 Let $f(x) = 4\sqrt{3}e^{-x}\cos x + 6e^{-x}$.

(1) Let a and b $(a < b)$ be the values of x satisfying $f(x) = 0$ on $0 \leqq x < 2\pi$. Then,

$$a = \frac{\boxed{\text{A}}}{\boxed{\text{B}}}\pi, \quad b = \frac{\boxed{\text{C}}}{\boxed{\text{D}}}\pi.$$

(2) The values of the constants p and q satisfying

$$\frac{d}{dx}\left(pe^{-x}\cos x + qe^{-x}\sin x\right) = e^{-x}\cos x$$

are given by

$$p = \frac{\boxed{\text{EF}}}{\boxed{\text{G}}}, \quad q = \frac{\boxed{\text{H}}}{\boxed{\text{I}}}.$$

(3) Using the values of a and b obtained in (1), we set $A = e^{-a}$ and $B = e^{-b}$. When we calculate the value of $\int_a^b f(x)\,dx$, we obtain

$$\int_a^b f(x)\,dx = \left(\boxed{\text{J}} - \sqrt{\boxed{\text{K}}}\right)A - \left(\boxed{\text{L}} + \sqrt{\boxed{\text{M}}}\right)B.$$

- memo -

Q 2 In the following questions, for **S** and **T**, choose the appropriate expression from among the choices ⓪ ～ ⑨ below.

Consider the definite integral $S = \int_0^a x\sqrt{\frac{1}{3}x+2}\ dx$.

(1) Set $t = \sqrt{\frac{1}{3}x+2}$. Then we have

$$\int x\sqrt{\frac{1}{3}x+2}\ dx = \boxed{\text{NO}} \int \left(t^{\boxed{\text{P}}} - \boxed{\text{Q}}\, t^{\boxed{\text{R}}}\right) dt$$

$$= \boxed{\text{S}} + C,$$

where C is the integral constant.

(2) Using the result in (1), we have

$$S = \boxed{\text{T}}.$$

Thus we obtain

$$\lim_{a\to\infty} \frac{S}{a^{\frac{\boxed{\text{U}}}{\boxed{\text{V}}}}} = \frac{\boxed{\text{W}}\sqrt{\boxed{\text{X}}}}{\boxed{\text{YZ}}}.$$

⓪ $\frac{6}{5}t^5(3t^2-10)$

① $\frac{6}{5}t^3(3t^2-10)$

② $\frac{12}{5}t^5(3t^2-5)$

③ $\frac{12}{5}t^3(3t^2-5)$

④ $\frac{6}{5}t^3(3t^2-5)$

⑤ $\frac{6}{5}\left\{\left(\sqrt{\frac{1}{3}a+2}\right)^5(a-4)+8\sqrt{2}\right\}$

⑥ $\frac{12}{5}\left\{\left(\sqrt{\frac{1}{3}a+2}\right)^3(a-2)+4\sqrt{2}\right\}$

⑦ $\frac{12}{5}\left\{\left(\sqrt{\frac{1}{3}a+2}\right)^5(a-2)+4\sqrt{2}\right\}$

⑧ $\frac{6}{5}\left\{\left(\sqrt{\frac{1}{3}a+2}\right)^3(a-4)+8\sqrt{2}\right\}$

⑨ $\frac{6}{5}\left\{\left(\sqrt{\frac{1}{3}a+2}\right)^3(a-2)+8\sqrt{2}\right\}$

- memo -

This is the end of the questions for Part IV.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.