Question Booklet

2015 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, $\cdots$ in the questions represents a numeral (from 0 to 9) or the minus sign ($-$). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$ is used later in the question. In such a case, the symbol is shaded when it is used later, as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$.

Note the following :

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example : Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\dfrac{\boxed{\text{A}}\sqrt{\boxed{\text{B}}}}{\boxed{\text{C}}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to $\boxed{\text{DE}}\,x$ is $-x$, mark "$-$" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label "Course 1" and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

<u>If you do not correctly fill in the appropriate oval, your answers will not be graded.</u>

I

Q 1 Let a and b be real numbers where $0 < b < 7$. Let us consider the maximum value M and the minimum value m of the quadratic function

$$f(x) = x^2 - 6x + a$$

over the interval $b \leqq x \leqq 7$.

The function $f(x)$ can be represented as

$$f(x) = (x - \boxed{\textsf{A}})^2 + a - \boxed{\textsf{B}}.$$

(1) For each of $\boxed{\textsf{C}}$ ～ $\boxed{\textsf{G}}$ in the following statements, choose the correct answer from among ⓪ ～ ⑨ below.

We are to find M and m. There are two cases.

(i) When $0 < b \leqq \boxed{\textsf{C}}$, then

$$M = \boxed{\textsf{D}}, \quad m = \boxed{\textsf{E}}.$$

(ii) When $\boxed{\textsf{C}} < b < 7$, then

$$M = \boxed{\textsf{F}}, \quad m = \boxed{\textsf{G}}.$$

⓪ 0	① 1	② 2	③ 3
④ $a - 6$	⑤ $a + 7$	⑥ $a + 8$	⑦ $a - 9$
⑧ $b^2 - 6b + a$	⑨ $b^2 + 6b + a$		

(2) In the case that $M = 13$ and $m = 1$, we have

$$a = \boxed{\textsf{H}}, \quad b = \boxed{\textsf{I}}.$$

- memo -

Q 2 Let us simultaneously throw three dice which are different in size and denote the number on the large, medium and small dice by x, y and z, respectively.

Let A be the event where $x = y = z$;
let B be the event where $x + y + z = 6$;
let C be the event where $x + y = z$.

(1) The numbers of outcomes in event A is $\boxed{\mathsf{J}}$, in event B is $\boxed{\mathsf{KL}}$, and in event C is $\boxed{\mathsf{MN}}$.

(2) The numbers of outcomes in event $A \cap B$ is $\boxed{\mathsf{O}}$, in event $B \cap C$ is $\boxed{\mathsf{P}}$, and in event $C \cap A$ is $\boxed{\mathsf{Q}}$.

(3) The probability of event $B \cup C$ is

$$P(B \cup C) = \frac{\boxed{\mathsf{RS}}}{\boxed{\mathsf{TUV}}}.$$

- memo -

This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{W}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

II

Q 1 Consider the expression in x

$$P = |x-1| + |x-2| + |x-a|.$$

We are to find the range of real numbers a such that the value of P is minimized at $x = a$.

First, let us note that the inequality

$$|x-1| + |x-2| + |x-a| \geqq |x-1| + |x-2|$$

always holds, and is an equality in the case $x = a$.

When we set

$$y = |x-1| + |x-2|, \quad \cdots\cdots\cdots \quad ①$$

we have

$$y = \begin{cases} -\boxed{\text{A}}\,x + \boxed{\text{B}} & (x < \boxed{\text{C}}) \\ \boxed{\text{D}} & (\boxed{\text{C}} \leqq x \leqq \boxed{\text{E}}) \\ \boxed{\text{F}}\,x - \boxed{\text{G}} & (\boxed{\text{E}} < x)\,. \end{cases}$$

When we consider the graph of ①, we see that the minimum value of y is $\boxed{\text{H}}$ and y takes the value $\boxed{\text{H}}$ at every x satisfying $\boxed{\text{I}} \leqq x \leqq \boxed{\text{J}}$.

Thus, for every a satisfying $\boxed{\text{K}} \leqq a \leqq \boxed{\text{L}}$, the value of P is minimized at $x = a$ and its value there is $\boxed{\text{M}}$.

- memo -

Q 2 Let a and b be natural numbers such that the greatest common divisor of a and b is 3. We are to find the natural numbers a and b such that

$$3a - 2b = \ell + 3 \qquad \cdots\cdots\cdots \quad ①$$

is satisfied, where ℓ is the least common multiple of a and b.

When we set $a = 3p$ and $b = 3q$, the natural numbers p and q are mutually prime (co-prime), and hence $\ell = \boxed{\mathsf{N}}\,pq$. Thus using p and q, the equality ① can be transformed to

$$pq - \boxed{\mathsf{O}}\,p + \boxed{\mathsf{P}}\,q + \boxed{\mathsf{Q}} = 0.$$

This can be further transformed to

$$(p + \boxed{\mathsf{R}})(q - \boxed{\mathsf{S}}) = -\boxed{\mathsf{T}}.$$

Among the pairs of integers p and q which satisfy this equation, the pair such that both a and b are natural numbers is

$$p = \boxed{\mathsf{U}}, \qquad q = \boxed{\mathsf{V}},$$

which gives

$$a = \boxed{\mathsf{WX}}, \qquad b = \boxed{\mathsf{Y}}.$$

- memo -

This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer space $\boxed{\textbf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

For each of $\boxed{\textbf{A}}$ ～ $\boxed{\textbf{M}}$ in the following statements, choose the correct answer from among ⓪ ～ ⑨ at the bottom of this page.

We are to solve the following simultaneous inequalities

$$\begin{cases} x^2 - 2x < 3 & \cdots\cdots\cdots \quad ① \\ ax^2 - ax - x + 1 > 0\ , & \cdots\cdots\cdots \quad ② \end{cases}$$

where $0 < a < 1$.

When we solve ①, we have

$$\boxed{\textbf{A}} < x < \boxed{\textbf{B}}.$$

Next, when we transform ②, we have

$$\left(ax - \boxed{\textbf{C}}\right)\left(x - \boxed{\textbf{D}}\right) > 0.$$

Hence, noting that $0 < a < 1$, we see that the solution of ② is

$$x < \boxed{\textbf{E}} \quad \text{or} \quad \boxed{\textbf{F}} < x.$$

Thus, when $0 < a \leqq \boxed{\textbf{G}}$, the solution of the simultaneous inequalities is

$$\boxed{\textbf{H}} < x < \boxed{\textbf{I}},$$

and when $\boxed{\textbf{G}} < a < 1$, the solution is

$$\boxed{\textbf{J}} < x < \boxed{\textbf{K}} \quad \text{or} \quad \boxed{\textbf{L}} < x < \boxed{\textbf{M}},$$

where $\boxed{\textbf{K}} < \boxed{\textbf{M}}$.

⓪ 0	① 1	② 2	③ 3	④ -1
⑤ $\frac{1}{2}$	⑥ $\frac{1}{3}$	⑦ $\frac{1}{a}$	⑧ $\frac{2}{a}$	⑨ $\frac{3}{a}$

- memo -

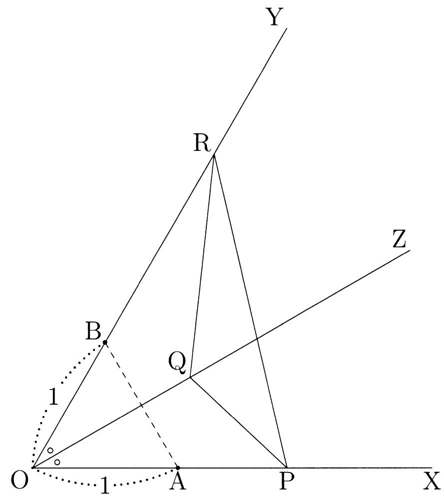
This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\mathbf{N}}$ ～ $\boxed{\mathbf{Z}}$ of Part $\boxed{\text{III}}$ blank.

IV

In the figure to the right, let $\angle XOY = 60°$, and let OZ be the half-line (ray) which bisects $\angle XOY$. In addition, the points A and B on the half-lines OX and OY satisfy $OA = OB = 1$.

Let P, Q and R be moving points on OX, OZ and OY that start simultaneously from A, O and B, moving in the direction away from point O at the speeds of 1, $\sqrt{3}$ and 2 units per second.

We are to find the moment at which the three points P, Q and R are arranged on a straight line by considering the area of the triangle PQR.



First, the lengths of OP, OQ and OR at t seconds after the start are

$$OP = t + \boxed{\mathbf{A}}, \quad OQ = \sqrt{\boxed{\mathbf{B}}}\,t, \quad OR = \boxed{\mathbf{C}}\,t + \boxed{\mathbf{D}}.$$

At this time the areas of the triangles are

$$\triangle OPQ = \frac{\sqrt{\boxed{\mathbf{E}}}\,t(t + \boxed{\mathbf{F}})}{4},$$

$$\triangle ORQ = \frac{\sqrt{\boxed{\mathbf{G}}}\,t(\boxed{\mathbf{H}}\,t + \boxed{\mathbf{I}})}{4},$$

$$\triangle OPR = \frac{\sqrt{\boxed{\mathbf{J}}}\,(t + \boxed{\mathbf{K}})(\boxed{\mathbf{L}}\,t + \boxed{\mathbf{M}})}{4}.$$

Hence we obtain

$$\triangle PQR = \frac{\sqrt{\boxed{\mathbf{N}}}}{4}\left|-t^2 + t + \boxed{\mathbf{O}}\right|.$$

So, to find the moment such that the three points P, Q and R are arranged on a straight line, we should find the case where

$$t^2 - t - \boxed{\mathbf{O}} = \boxed{\mathbf{P}}.$$

Thus the time required is

$$t = \frac{\boxed{\mathbf{Q}} + \sqrt{\boxed{\mathbf{R}}}}{\boxed{\mathbf{S}}} \text{ (seconds)}.$$

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This is the end of the questions for Part $\boxed{\text{IV}}$.

Leave the answer spaces $\boxed{\mathsf{T}}$ ～ $\boxed{\mathsf{Z}}$ of Part $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

＜ Example ＞

解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

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⓪ 0	① 1	② 2	③ 3
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(2) In the case that $M = 13$ and $m = 1$, we have

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- memo -

Q 2 Let us simultaneously throw three dice which are different in size and denote the number on the large, medium and small dice by x, y and z, respectively.

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(3) The probability of event $B \cup C$ is

$$P(B \cup C) = \frac{\boxed{\textsf{RS}}}{\boxed{\textsf{TUV}}}.$$

- memo -

This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{W}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

II

Q 1 Given two points A$(1,-1,0)$ and B$(-2,1,2)$ in a coordinate space with the origin O, let us set $\overrightarrow{\mathrm{OA}}=\vec{a}$, $\overrightarrow{\mathrm{OB}}=\vec{b}$.

(1) First, we are to find the value of t at which $\left|\vec{a}+t\vec{b}\right|$ is minimized. Since

$$\left|\vec{a}+t\vec{b}\right|^2=\boxed{\mathbf{A}}\,t^2-\boxed{\mathbf{B}}\,t+\boxed{\mathbf{C}},$$

$\left|\vec{a}+t\vec{b}\right|$ is minimized at $t=\dfrac{\boxed{\mathbf{D}}}{\boxed{\mathbf{E}}}$, and its minimum value is $\boxed{\mathbf{F}}$.

(2) Next, the vectors $\vec{c}$ which are orthogonal to the vectors $\vec{a}$ and $\vec{b}$ can be represented as

$$\vec{c}=s\left(\boxed{\mathbf{G}},\ \boxed{\mathbf{H}},\ 1\right)$$

where s is a non-zero real number.

Now, let C and D be the points such that $\overrightarrow{\mathrm{OC}}=\left(\boxed{\mathbf{G}},\ \boxed{\mathbf{H}},\ 1\right)$ and $\overrightarrow{\mathrm{OD}}=3\vec{a}+\vec{b}$.

Since $\angle\mathrm{CBD}=\dfrac{\pi}{\boxed{\mathbf{I}}}$, the area of the triangle BCD is $\dfrac{\boxed{\mathbf{J}}\sqrt{\boxed{\mathbf{K}}}}{\boxed{\mathbf{L}}}$.

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Q 2 Let us consider the solutions to the equation in the complex number z

$$z^4 = -324 \quad \cdots\cdots\cdots \quad ①$$

and the solutions to the equation in the complex number z

$$z^4 = t^4 \quad \cdots\cdots\cdots \quad ②$$

where t is a positive real number.

(1) To find the solutions to ①, let us set

$$z = r(\cos\theta + i\sin\theta) \qquad (r > 0,\ 0 < \theta \leqq 2\pi).$$

Then

$$z^4 = r^{\boxed{\mathsf{M}}}(\cos \boxed{\mathsf{N}}\,\theta + i\sin \boxed{\mathsf{N}}\,\theta).$$

The values of r and θ such that this expression is equal to -324 are

$$r = \boxed{\mathsf{O}}\sqrt{\boxed{\mathsf{P}}},$$

$$\theta = \frac{\boxed{\mathsf{Q}}}{\boxed{\mathsf{R}}}\pi,\ \frac{\boxed{\mathsf{S}}}{\boxed{\mathsf{R}}}\pi,\ \frac{\boxed{\mathsf{T}}}{\boxed{\mathsf{R}}}\pi,\ \frac{\boxed{\mathsf{U}}}{\boxed{\mathsf{R}}}\pi,$$

where $\boxed{\mathsf{Q}} < \boxed{\mathsf{S}} < \boxed{\mathsf{T}} < \boxed{\mathsf{U}}$.

(2) There are $\boxed{\mathsf{V}}$ solutions to equation ②, and these solutions are dependent on t. Now, consider any one of the solutions to ① and any one of the solutions to ②, and let d be the distance on the complex number plane between these two solutions. Then, over the interval $0 < t \leqq 4$, the minimum value of d is $\boxed{\mathsf{W}}$ and the maximum value is $\sqrt{\boxed{\mathsf{XY}}}$.

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This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer space $\boxed{\textbf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

Let $f(x) = x^4 + 2x^3 - 12x^2 + 4$. We are to find the values of p such that we can draw three tangents to the curve $y = f(x)$ from the point $\mathrm{P}(0, p)$ on the y-axis.

(i) The equation of the tangent to the curve $y = f(x)$ at the point $\bigl(t, f(t)\bigr)$ is

$$y = \Bigl(\boxed{\mathsf{A}}\,t^3 + \boxed{\mathsf{B}}\,t^2 - \boxed{\mathsf{CD}}\,t\Bigr)x - \boxed{\mathsf{E}}\,t^4 - \boxed{\mathsf{F}}\,t^3 + \boxed{\mathsf{GH}}\,t^2 + \boxed{\mathsf{I}}\,.$$

The condition under which this straight line passes through the point $\mathrm{P}(0, p)$ is that

$$p = -\boxed{\mathsf{J}}\,t^4 - \boxed{\mathsf{K}}\,t^3 + \boxed{\mathsf{LM}}\,t^2 + \boxed{\mathsf{N}}\,. \qquad \cdots\cdots\cdots \quad ①$$

(ii) For $\boxed{\mathsf{O}}$ and $\boxed{\mathsf{S}}$ in the following statements, choose either ⓪ or ① and for the other $\boxed{\phantom{\mathsf{X}}}$, enter the correct number.

⓪ local minimum ① local maximum

When the right side of ① is set to $g(t)$, the function $g(t)$ takes a $\boxed{\mathsf{O}}$ at $t = \boxed{\mathsf{PQ}}$ and $t = \boxed{\mathsf{R}}$. On the other hand, $g(t)$ takes a $\boxed{\mathsf{S}}$ at $t = \boxed{\mathsf{T}}$.

Hence the values of p such that we can draw three tangents to the curve $y = f(x)$ from the point $\mathrm{P}(0, p)$ are

$$p = \boxed{\mathsf{U}} \quad \text{and} \quad p = \boxed{\mathsf{V}},$$

where $\boxed{\mathsf{U}} < \boxed{\mathsf{V}}$.

- memo -

This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\textbf{W}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{III}}$ blank.

IV

The coordinates (x, y) of a moving point P are given by the following functions in time t:

$$x = 4t - \sin 4t,$$
$$y = 4 - \cos 4t.$$

(1) The derivatives of x and y with respect to t are

$$\frac{dx}{dt} = \boxed{\mathsf{A}}\left(\boxed{\mathsf{B}} - \cos 4t\right),$$
$$\frac{dy}{dt} = \boxed{\mathsf{C}}\ \sin 4t.$$

Hence we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \boxed{\mathsf{DE}}\ \sin^2 \boxed{\mathsf{F}}\, t.$$

(2) As the point P moves from the time $t = 0$ to the time $t = 2\pi$, its speed v is maximized a total of $\boxed{\mathsf{G}}$ times. Let us denote by t_0 the moment of the first time the speed is maximized and the moment of the last time it is maximized by t_1. Then

$$t_0 = \frac{\boxed{\mathsf{H}}}{\boxed{\mathsf{I}}}\pi, \quad t_1 = \frac{\boxed{\mathsf{J}}}{\boxed{\mathsf{K}}}\pi,$$

and the maximum speed is $v = \boxed{\mathsf{L}}$.

(3) For t_0 and t_1 in (2), the distance that point P moves during the period from $t = t_0$ to $t = t_1$ is $\boxed{\mathsf{MN}}$.

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This is the end of the questions for Part $\boxed{\text{IV}}$.

Leave the answer spaces $\boxed{\mathbf{O}}$ ～ $\boxed{\mathbf{Z}}$ of Part $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.