

# Mathematics (80 min.)

## 【Course 1(Basic), Course 2(Advanced)】

※ Choose one of these courses and answer its questions only.

### I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

### II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

### III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ... in the questions represents a numeral (from 0 to 9) or the minus sign(-). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet(mark-sheet).
3. Sometimes an answer such as **A** or **BC** is used later in the question. In such a case, the symbol is shaded when it is used later, as **A** or **BC**.

#### Note the following :

- (1) Reduce square roots ( $\sqrt{\quad}$ ) as much as possible.  
(Example: Express  $\sqrt{32}$  as  $4\sqrt{2}$ , not as  $2\sqrt{8}$  or  $\sqrt{32}$ .)
- (2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.

(Example: Substitute  $\frac{1}{3}$  for  $\frac{2}{6}$ . Also simplify as follows:

$$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}. \text{ Then apply } \frac{-\sqrt{6}}{3} \text{ to the answer.})$$

- (3) If your answer to  $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$  is  $-\frac{\sqrt{3}}{4}$ , mark as shown below.
- (4) If the answer to  $\boxed{DE}x$  is  $-x$ , mark “-” for **D** and “1” for **E** as shown below.

<b>A</b>	●	0	1	2	3	4	5	6	7	8	9
<b>B</b>	⊖	0	1	2	●	4	5	6	7	8	9
<b>C</b>	⊖	0	1	2	3	●	5	6	7	8	9
<b>D</b>	●	0	1	2	3	4	5	6	7	8	9
<b>E</b>	⊖	0	●	2	3	4	5	6	7	8	9

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*				*				
Name											



# Mathematics Course 1

(Basic Course)

**(Course 2 begins on page 15)**

## Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

**If you do not correctly fill in the appropriate oval, your answers will not be graded.**

Mathematics—2

I

Q 1 The quadratic function  $f(x) = 2x^2 + ax - 1$  in  $x$  satisfies

$$f(-1) \geq -3, \quad f(2) \geq 3. \quad \dots\dots\dots \textcircled{1}$$

Let us consider the minimum value  $m$  of  $f(x)$ .

(1)  $m$  can be expressed in terms of  $a$  as

$$m = -\frac{\boxed{\text{A}}}{\boxed{\text{B}}} a^2 - \boxed{\text{C}}.$$

(2) The range of the values of  $a$  such that  $f(x)$  satisfies condition  $\textcircled{1}$  is

$$\boxed{\text{DE}} \leq a \leq \boxed{\text{F}}.$$

(3) The value of  $m$  is maximized when the axis of symmetry of the graph of  $y = f(x)$  is the straight line  $x = \boxed{\text{G}}$ , and then the value of  $m$  is  $\boxed{\text{HI}}$ .

(4) The value of  $m$  is minimized when the axis of symmetry of the graph of  $y = f(x)$  is the straight line  $x = \boxed{\text{JK}}$ , and then the value of  $m$  is  $\boxed{\text{LM}}$ .

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## Mathematics—4

**Q 2** A triangle  $ABC$  is drawn on a plane, and a ball is placed on vertex  $A$ . A dice is rolled, and the ball is moved according to the following rules:

- (i) when the ball is on  $A$ , if the number on the dice is 1 the ball is moved to  $B$ , otherwise it stays on  $A$ ;
- (ii) when the ball is on  $B$ , if the number on the dice is less than or equal to 4 the ball is moved to  $C$ , otherwise it stays on  $B$ .

If the ball is moved to  $C$ , the trials are stopped.

We are to find the probability that the ball is moved to  $C$  within 4 rolls of the dice.

(1) The probability that the ball is moved to  $C$  on the second roll of the dice is  $\frac{1}{\boxed{N}}$ .

(2) The probability that the ball is moved to  $C$  on the third roll of the dice is  $\frac{\boxed{O}}{\boxed{PQ}}$ .

(3) The probability that the ball is moved to  $C$  on the fourth roll of the dice is  $\frac{\boxed{RS}}{\boxed{TUV}}$ .

Therefore, the probability that the ball is moved to  $C$  within 4 rolls of the dice is  $\frac{\boxed{WX}}{\boxed{YZ}}$ .

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This is the end of the questions for Part I.

II

**Q 1** Let  $a$  and  $b$  be rational numbers and let  $p$  be a real number. Consider the quadratic equation

$$x^2 + ax + b = 0 \quad \dots\dots\dots \textcircled{1}$$

which has a solution  $x = \frac{\sqrt{5} + 3}{\sqrt{5} + 2}$ , and consider the inequality

$$x + 1 < 2x + p + 3. \quad \dots\dots\dots \textcircled{2}$$

(1) First, we are to find the values of  $a$  and  $b$ .

When we rationalize the denominator of  $x = \frac{\sqrt{5} + 3}{\sqrt{5} + 2}$ , we have

$$x = \sqrt{\boxed{\text{A}}} - \boxed{\text{B}}.$$

Since this is a solution of equation  $\textcircled{1}$ , by substituting this in  $\textcircled{1}$  we have

$$-a + b + \boxed{\text{C}} + (a - \boxed{\text{D}})\sqrt{\boxed{\text{E}}} = 0.$$

Hence we see that

$$a = \boxed{\text{F}} \quad \text{and} \quad b = \boxed{\text{GH}}.$$

(2) Next, we are to find the smallest integer  $p$  such that both solutions of equation  $\textcircled{1}$  satisfy inequality  $\textcircled{2}$ .

When we solve inequality  $\textcircled{2}$ , we have

$$x > -p - \boxed{\text{I}}.$$

Since both solutions of equation  $\textcircled{1}$  satisfy this, we see that

$$p > \sqrt{\boxed{\text{J}}} - \boxed{\text{K}}.$$

Hence the smallest integer  $p$  is  $\boxed{\text{L}}$ .



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## Mathematics—8

**Q 2** Consider the quadratic function

$$f(x) = \frac{3}{4}x^2 - 3x + 4.$$

Let  $a$  and  $b$  be real numbers satisfying  $0 < a < b$  and  $2 < b$ . We are to find the values of  $a$  and  $b$  such that the range of the values of the function  $y = f(x)$  on  $a \leq x \leq b$  is  $a \leq y \leq b$ .

Since the equation of the axis of symmetry of the graph of  $y = f(x)$  is  $x = \boxed{\text{M}}$ , we divide the problem into two cases as follows:

(i)  $\boxed{\text{M}} \leq a$ ;

(ii)  $0 < a < \boxed{\text{M}}$ .

In the case of (i), since the values of  $f(x)$  increase with  $x$  on  $a \leq x \leq b$ , the equations  $f(a) = a$  and  $f(b) = b$  have to be satisfied. By solving these, we obtain  $a = \frac{\boxed{\text{N}}}{\boxed{\text{O}}}$  and  $b = \boxed{\text{P}}$ . However, this  $a$  does not satisfy (i).

In the case of (ii), since the minimum value of  $f(x)$  on  $a \leq x \leq b$  is  $\boxed{\text{Q}}$ , we have

$$a = \boxed{\text{R}}.$$

This satisfies (ii).

Then since  $f(a) = \frac{\boxed{\text{S}}}{\boxed{\text{T}}} < b$ , we have  $f(b) = b$ . Hence, we obtain

$$b = \boxed{\text{U}}.$$

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This is the end of the questions for Part **II**. Leave the answer spaces **V** ~ **Z** of Part **II** blank.

III

Consider four natural numbers  $a, b, c$  and  $d$  satisfying  $1 < a < b < c < d$ . Suppose that two sets using these numbers,  $A = \{a, b, c, d\}$  and  $B = \{a^2, b^2, c^2, d^2\}$ , satisfy the following two conditions:

- (i) Just two elements belong to the intersection  $A \cap B$ , and the sum of these two elements is greater than or equal to 15, and less than or equal to 25.
- (ii) The sum of all the elements belonging to the union  $A \cup B$ , is less than or equal to 300.

We are to find the values of  $a, b, c$  and  $d$ .

First, set  $A \cap B = \{x, y\}$ , where  $x < y$ . Since  $x \in B$  and  $y \in B$ , it follows from (i) that  $y = \boxed{\text{AB}}$  and that  $x$  is either  $\boxed{\text{C}}$  or  $\boxed{\text{D}}$ . (Write the answers in the order  $\boxed{\text{C}} < \boxed{\text{D}}$ .) Here, when we consider (ii), we see that  $x = \boxed{\text{E}}$ . Hence  $A$  includes the elements  $\boxed{\text{F}}$ ,  $\boxed{\text{F}}^2$  and  $\boxed{\text{F}}^4$ .

Furthermore, when we denote the remaining element of  $A$  by  $z$ , from (ii) we see that  $z$  satisfies

$$z^2 + z \leq \boxed{\text{GH}}.$$

Hence we have  $z = \boxed{\text{I}}$ .

From the above we obtain

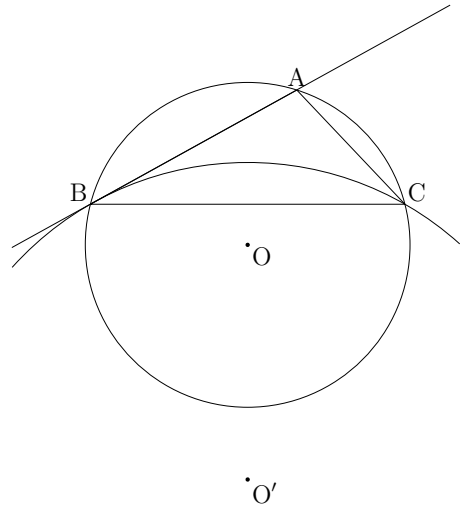
$$a = \boxed{\text{J}}, \quad b = \boxed{\text{K}}, \quad c = \boxed{\text{L}} \quad \text{and} \quad d = \boxed{\text{MN}}.$$

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This is the end of the questions for Part . Leave the answer spaces  ~  of Part  blank.

IV

Let the lengths of the three sides of the triangle ABC be  $AB = 6$ ,  $BC = 8$  and  $CA = 4$ . Let  $O'$  be the center of the circle which passes through the two points B and C and is tangent to the straight line AB. Let O be the center of the circle circumscribed about triangle ABC. We are to find the length of the line segment  $OO'$ .



(1) First, we have  $\cos \angle ABC = \frac{\boxed{A}}{\boxed{B}}$  and  $\sin \angle ABC = \frac{\sqrt{\boxed{CD}}}{\boxed{E}}$ .

(2) The radius of the circle circumscribed about triangle ABC is  $\frac{\boxed{FG} \sqrt{\boxed{HI}}}{\boxed{JK}}$ .

(3) When the intersection point of the straight line  $OO'$  and the side BC is denoted by D, we have

$$OD = \frac{\boxed{L} \sqrt{\boxed{MN}}}{\boxed{OP}} \quad \text{and} \quad O'D = \frac{\boxed{QR} \sqrt{\boxed{ST}}}{\boxed{UV}}.$$

Thus we have  $OO' = \frac{\boxed{W} \sqrt{\boxed{XY}}}{\boxed{Z}}$ .

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This is the end of the questions for Part IV.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

**Please check once more that you have properly marked your course number  
as “Course 1” on your answer sheet.**

**Do not take this question booklet out of the room.**





# Mathematics Course 2

(Advanced Course)

## Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	<div style="border: 1px solid black; border-radius: 50%; width: 40px; height: 40px; margin: 0 auto; display: flex; align-items: center; justify-content: center;">           コース 2 Course 2         </div>
○	●

**If you do not correctly fill in the appropriate oval, your answers will not be graded.**

I

**Q 1** The quadratic function  $f(x) = 2x^2 + ax - 1$  in  $x$  satisfies

$$f(-1) \geq -3, \quad f(2) \geq 3. \quad \dots\dots\dots \textcircled{1}$$

Let us consider the minimum value  $m$  of  $f(x)$ .

(1)  $m$  can be expressed in terms of  $a$  as

$$m = -\frac{\boxed{\text{A}}}{\boxed{\text{B}}} a^2 - \boxed{\text{C}}.$$

(2) The range of the values of  $a$  such that  $f(x)$  satisfies condition  $\textcircled{1}$  is

$$\boxed{\text{DE}} \leq a \leq \boxed{\text{F}}.$$

(3) The value of  $m$  is maximized when the axis of symmetry of the graph of  $y = f(x)$  is the straight line  $x = \boxed{\text{G}}$ , and then the value of  $m$  is  $\boxed{\text{HI}}$ .

(4) The value of  $m$  is minimized when the axis of symmetry of the graph of  $y = f(x)$  is the straight line  $x = \boxed{\text{JK}}$ , and then the value of  $m$  is  $\boxed{\text{LM}}$ .

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## Mathematics—18

**Q 2** A triangle  $ABC$  is drawn on a plane, and a ball is placed on vertex  $A$ . A dice is rolled, and the ball is moved according to the following rules:

- (i) when the ball is on  $A$ , if the number on the dice is 1 the ball is moved to  $B$ , otherwise it stays on  $A$ ;
- (ii) when the ball is on  $B$ , if the number on the dice is less than or equal to 4 the ball is moved to  $C$ , otherwise it stays on  $B$ .

If the ball is moved to  $C$ , the trials are stopped.

We are to find the probability that the ball is moved to  $C$  within 4 rolls of the dice.

(1) The probability that the ball is moved to  $C$  on the second roll of the dice is  $\frac{1}{\boxed{\text{N}}}$ .

(2) The probability that the ball is moved to  $C$  on the third roll of the dice is  $\frac{\boxed{\text{O}}}{\boxed{\text{PQ}}}$ .

(3) The probability that the ball is moved to  $C$  on the fourth roll of the dice is  $\frac{\boxed{\text{RS}}}{\boxed{\text{TUV}}}$ .

Therefore, the probability that the ball is moved to  $C$  within 4 rolls of the dice is  $\frac{\boxed{\text{WX}}}{\boxed{\text{YZ}}}$ .

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This is the end of the questions for Part I.

II

**Q 1** We are to find the general term  $a_n$  of the sequence  $\{a_n\}$  which is determined by the recurrence formula

$$a_1 = 18, \quad a_{n+1} - 12a_n + 3^{n+2} = 0 \quad (n = 1, 2, 3, \dots).$$

When we define a sequence  $\{b_n\}$  by

$$b_n = \frac{a_n}{\boxed{\mathbf{A}}^n} \quad (n = 1, 2, 3, \dots),$$

$\{b_n\}$  satisfies

$$b_1 = \boxed{\mathbf{B}}, \quad b_{n+1} - \boxed{\mathbf{C}}b_n + \boxed{\mathbf{D}} = 0 \quad (n = 1, 2, 3, \dots).$$

This recurrence formula can be transformed into

$$b_{n+1} - \boxed{\mathbf{E}} = \boxed{\mathbf{F}} (b_n - \boxed{\mathbf{E}}).$$

Next, when we define a sequence  $\{c_n\}$  by

$$c_n = b_n - \boxed{\mathbf{E}} \quad (n = 1, 2, 3, \dots),$$

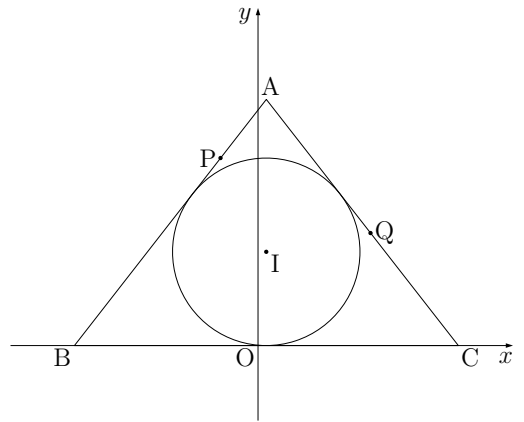
$\{c_n\}$  is a geometric progression such that the first term is  $\boxed{\mathbf{G}}$  and the common ratio is  $\boxed{\mathbf{H}}$ .

Hence we have

$$a_n = \boxed{\mathbf{I}}^n \left( \boxed{\mathbf{J}} \cdot \boxed{\mathbf{K}}^{n-1} + \boxed{\mathbf{L}} \right) \quad (n = 1, 2, 3, \dots).$$

- memo -

**Q 2** As shown in the figure to the right, on an  $xy$ -plane whose origin is  $O$ , let us consider an isosceles triangle  $ABC$  satisfying  $AB = AC$ . Furthermore, suppose that side  $AB$  passes through  $P(-1, 5)$  and side  $AC$  passes through  $Q(3, 3)$ .



Let us consider the radius of the inscribed circle of the triangle  $ABC$ .

Denote the straight line passing through the two points  $A$  and  $B$  by  $\ell_1$  and the straight line passing through the two points  $A$  and  $C$  by  $\ell_2$ . When we denote the slope of  $\ell_1$  by  $a$ , the equations of  $\ell_1$  and  $\ell_2$  are

$$\ell_1 : y = ax + a + \boxed{\text{M}},$$

$$\ell_2 : y = -ax + \boxed{\text{N}}a + \boxed{\text{O}}.$$

Denote the center and the radius of the inscribed circle by  $I$  and  $r$ , respectively. Then the coordinates of  $I$  are  $\left( \boxed{\text{P}} - \frac{\boxed{\text{Q}}}{a}, r \right)$ .

Hence  $r$  can be expressed in terms of  $a$  as

$$r = \frac{\boxed{\text{R}}a + \boxed{\text{S}}}{\boxed{\text{T}} + \sqrt{a^2 + \boxed{\text{U}}}}.$$

In particular, when  $r = \frac{5}{2}$ , the coordinates of vertex  $A$  are  $\left( \frac{\boxed{\text{V}}}{\boxed{\text{W}}}, \frac{\boxed{\text{XY}}}{\boxed{\text{Z}}} \right)$ .



- memo -

This is the end of the questions for Part II.

**III**

We are to find the range of the values of  $k$  such that the inequality

$$\frac{\log 3x}{4x + 1} \leq \log \left( \frac{2kx}{4x + 1} \right) \quad \dots\dots\dots \textcircled{1}$$

holds for all positive real numbers  $x$ , where  $\log$  is the natural logarithm.

- (1) For **A** and **B** in the following sentences, choose the correct answer from among  $\textcircled{0} \sim \textcircled{8}$  below.

By transforming inequality  $\textcircled{1}$  we obtain

$$\log k \geq \text{A} \quad \dots\dots\dots \textcircled{2}$$

Here, when the right side of  $\textcircled{2}$  is denoted by  $g(x)$  and this  $g(x)$  is differentiated with respect to  $x$ , we have

$$g'(x) = \text{B}$$

- |  |   |
|--|---|
| <p><math>\textcircled{0} \quad \frac{\log 3x}{4x + 1} - \log(4x + 1) - \log 2x</math></p> <p><math>\textcircled{2} \quad \frac{\log 3x}{4x + 1} + \log(4x + 1) + \log 2x</math></p> <p><math>\textcircled{4} \quad \frac{4 \log 3x}{(4x + 1)^2}</math></p> <p><math>\textcircled{6} \quad -\frac{4 \log 3x}{(4x + 1)^2}</math></p> <p><math>\textcircled{8} \quad -\frac{3 \log 2x}{(4x + 1)^2}</math></p> | <p><math>\textcircled{1} \quad \frac{\log 3x}{4x + 1} - \log(4x + 1) + \log 2x</math></p> <p><math>\textcircled{3} \quad \frac{\log 3x}{4x + 1} + \log(4x + 1) - \log 2x</math></p> <p><math>\textcircled{5} \quad \frac{3x + 2 + \log 3x}{(4x + 1)^2}</math></p> <p><math>\textcircled{7} \quad \frac{3x - 2 - \log 2x}{(4x + 1)^2}</math></p> |
|--|---|

(This question is continued on the next page.)

(2) In the following sentences, for  $\boxed{\text{E}}$ ,  $\boxed{\text{F}}$  and  $\boxed{\text{G}}$ , choose the correct answer from among ① ~ ③ below. For the other  $\boxed{\phantom{\text{E}}}$ , enter the correct number.

Over the interval  $0 < x < \frac{\boxed{\text{C}}}{\boxed{\text{D}}}$ ,  $g(x)$  is  $\boxed{\text{E}}$  and over the interval  $\frac{\boxed{\text{C}}}{\boxed{\text{D}}} < x$ ,  $g(x)$  is  $\boxed{\text{F}}$ . Hence at  $x = \frac{\boxed{\text{C}}}{\boxed{\text{D}}}$ ,  $g(x)$  is  $\boxed{\text{G}}$ .

From the above, the range of the value of  $k$  such that inequality ① holds for all positive real numbers  $x$  is

$$k \geq \frac{\boxed{\text{H}}}{\boxed{\text{I}}}.$$

① increasing      ② decreasing      ③ maximized      ④ minimized

This is the end of the questions for Part  $\boxed{\text{III}}$ . Leave the answer spaces  $\boxed{\text{J}} \sim \boxed{\text{Z}}$  of Part  $\boxed{\text{III}}$  blank.

**IV**

Consider the following two curves

$$x^2 + y^2 = 1, \quad \dots\dots\dots \textcircled{1}$$

$$4xy = 1, \quad \dots\dots\dots \textcircled{2}$$

where  $x > 0, y > 0$ . We are to find the area  $S$  of the region bounded by curve  $\textcircled{1}$  and curve  $\textcircled{2}$ .

- (1) First, let P and Q be the intersection points of curves  $\textcircled{1}$  and  $\textcircled{2}$ , and let us denote the  $x$ -coordinates of P and Q by  $p$  and  $q$  ( $p < q$ ), respectively.

From  $\textcircled{1}$ , the coordinates  $(x, y)$  of the intersection points of curves  $\textcircled{1}$  and  $\textcircled{2}$  can be expressed as  $x = \cos \theta, y = \sin \theta$  ( $0 < \theta < \frac{\pi}{2}$ ). Then from  $\textcircled{2}$  we have

$$\sin \boxed{\text{A}} \theta = \frac{\boxed{\text{B}}}{\boxed{\text{C}}}.$$

From this we know that

$$\theta = \frac{\boxed{\text{D}}}{\boxed{\text{EF}}} \pi \quad \text{or} \quad \frac{\boxed{\text{G}}}{\boxed{\text{HI}}} \pi.$$

(Write the answers in the order such that  $\frac{\boxed{\text{D}}}{\boxed{\text{EF}}} < \frac{\boxed{\text{G}}}{\boxed{\text{HI}}}$ .)

Hence we have

$$p = \cos \frac{\boxed{\text{J}}}{\boxed{\text{KL}}} \pi, \quad q = \cos \frac{\boxed{\text{M}}}{\boxed{\text{NO}}} \pi.$$

(This question is continued on the next page.)

(2) Now we can find the value of  $S$ . Since

$$S = \int_p^q \left( \sqrt{1-x^2} - \frac{1}{4x} \right) dx,$$

we have to find the values of

$$I = \int_p^q \sqrt{1-x^2} dx, \quad J = \int_p^q \frac{1}{x} dx.$$

For  $I$ , when we set  $x = \cos \theta$  and calculate by substituting it for  $x$  in the integral, we have

$$I = \frac{\boxed{\text{P}}}{\boxed{\text{Q}}} \pi.$$

For  $J$ , we have

$$J = \log \left( \boxed{\text{R}} + \sqrt{\boxed{\text{S}}} \right),$$

where  $\log$  is the natural logarithm.

From these, we obtain

$$S = \frac{\boxed{\text{P}}}{\boxed{\text{Q}}} \pi - \frac{\boxed{\text{T}}}{\boxed{\text{U}}} \log \left( \boxed{\text{R}} + \sqrt{\boxed{\text{S}}} \right).$$

This is the end of the questions for Part  $\boxed{\text{IV}}$ .

Leave the answer spaces  $\boxed{\text{V}} \sim \boxed{\text{Z}}$  of Part  $\boxed{\text{IV}}$  blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part  $\boxed{\text{V}}$  blank.

**Please check once more that you have properly marked your course number as “Course 2” on your answer sheet.**

**Do not take this question booklet out of the room.**