Question Booklet

2013 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter A, B, C, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). Completely black out your answer for each letter in the corresponding line of the answer sheet (mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such as a case, the symbol is shaded when it is used later, as [A] or [BC].

Note the following :

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example : Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to [DE] x is $-x$, mark "−" for D and "1" for E as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*					
Name													

Mathematics Course 1

(Basic Course)

（Course 2 begins on page 15）

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely black out the oval under the label on your answer sheet as shown in the example on the right.

＜ Example ＞

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

<u>**If you do not correctly black out the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Suppose we have a quadratic function $y = ax^2 + bx + c$ in x which satisfies the following conditions【∗】:

【∗】 When $x = -1$, then $y = -8$ and when $x = 3$, then $y = 16$. Further, in the interval $-1 \leqq x \leqq 3$, the value of y increases with the increase of the value of x.

We are to find the conditions which a, b and c must satisfy.

From【∗】, it follows that b and c can be expressed in terms of a as

$$b = \boxed{\mathbf{AB}}\,a + \boxed{\mathbf{C}} \quad \cdots\cdots\cdots \text{①}$$

$$c = \boxed{\mathbf{DE}}\,a - \boxed{\mathbf{F}}. \quad \cdots\cdots\cdots \text{②}$$

Hence, the axis of symmetry of the graph of this quadratic function has the equation

$$x = \boxed{\mathbf{G}} - \frac{\boxed{\mathbf{H}}}{a}.$$

Thus a, b and c must satisfy the relationships ① and ②, and furthermore

$$0 < a \leqq \frac{\boxed{\mathbf{I}}}{\boxed{\mathbf{J}}} \quad \text{or} \quad \frac{\boxed{\mathbf{KL}}}{\boxed{\mathbf{M}}} \leqq a < 0.$$

- memo -

Q 2 Let a, b, c and d be real numbers satisfying $a < b < c < d$. Suppose that the two subsets of real numbers

$$A = \{x \mid a \leqq x \leqq c\}, \quad B = \{x \mid b \leqq x \leqq d\}$$

satisfy

$$A \cap B = \{x \mid x^2 - 4x + 3 \leqq 0\}.$$

Then, answer the questions for cases (1) and (2).

(1) Let the union of A and B be

$$A \cup B = \{x \mid x^2 - 5x - 24 \leqq 0\}.$$

Then the values of a, b, c and d are

$a =$ **NO**, $b =$ **P**, $c =$ **Q**, $d =$ **R**.

(2) Let the intersection of A and the complement $\overline{B}$ of B be

$$A \cap \overline{B} = \{x \mid x^2 + 5x - 6 \leqq 0 \text{ and } x \neq 1\},$$

and let the intersection of the complement $\overline{A}$ of A and B be

$$\overline{A} \cap B = \{x \mid x^2 - 9x + 18 \leqq 0 \text{ and } x \neq 3\}.$$

Then the values of a, b, c and d are

$a =$ **ST**, $b =$ **U**, $c =$ **V**, $d =$ **W**.

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This is the end of the questions for Part I. Leave the answer spaces **X** ～ **Z** of Part I blank.

II

Q 1 Consider a polynomial in x and y

$$P = (3x + 4y + 1)^5.$$

Let us denote the coefficient of $x^n y$ in the expansion of P by a_n, where n is an integer. Note that $x^0 = y^0 = 1$.

(1) Let us find the value of the coefficient a_1. First, we note that

$$P = \{(3x + 1) + 4y\}^5$$

and use the binomial theorem to expand P. Then xy appears when we expand the term $\boxed{\mathsf{AB}}(3x+1)^{\boxed{\mathsf{C}}}y$. Further, the coefficient for x in the expansion of $(3x+1)^{\boxed{\mathsf{C}}}$ is $\boxed{\mathsf{DE}}$. It follows that

$$a_1 = \boxed{\mathsf{FGH}}.$$

(2) The number of values which n can take is $\boxed{\mathsf{I}}$ in all. Also, the value of a_n is maximized at $n = \boxed{\mathsf{J}}$.

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Q 2 Consider the integral expression

$$P = (x-1)^2(y+5)+(2x-3)(y+4)-(x-1)^2.$$

(1) P can be transformed into

$$P = \left(x^2 - \boxed{\textsf{K}}\right)\left(y + \boxed{\textsf{L}}\right).$$

(2) The pairs (x, y) of integers x and y which give $P = 7$ are

$$\left(\pm\boxed{\textsf{M}}, \boxed{\textsf{NOP}}\right), \quad \left(\pm\boxed{\textsf{Q}}, \boxed{\textsf{RS}}\right).$$

(3) Let a be a rational number. If $x = \sqrt{2} + 2\sqrt{3}$ and $y = a + \sqrt{6}$, then the value of a such that the value of P is a rational number is $\boxed{\textsf{TU}}$.

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This is the end of the questions for Part II. Leave the answer spaces V ~ Z of Part II blank.

III

For each of **A** ～ **D** in questions (1) ～ (4) below, choose the appropriate answer from among ⓪ ～ ③ of each question. For **E** ～ **G** in question (5), put the correct number.

Suppose that a, b and c are integers, and $a > 0$. Also, suppose that the graph of a quadratic function $y = ax^2 - 2bx + c$ intersects the x-axis and all points of intersection are in the interval $0 < x < 1$.

(1) The relationship between a and b is **A**.

⓪ $a > b$　　① $a < b$

② $a = b$　　③ indeterminate

(2) The conditions on b and c are **B**.

⓪ $b < 0,\ c < 0$　　① $b < 0,\ c > 0$

② $b > 0,\ c < 0$　　③ $b > 0,\ c > 0$

(3) The relationship between $2b$ and $a + c$ is **C**.

⓪ $2b > a + c$　　① $2b < a + c$

② $2b = a + c$　　③ indeterminate

(4) The relationship between b and c is **D**.

⓪ $b > c$　　① $b < c$

② $b = c$　　③ indeterminate

(5) The smallest integer which a can take is **E**. In this case, the value of b is **F**, and the value of c is **G**.

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This is the end of the questions for Part III. Leave the answer spaces **H** ～ **Z** of Part III blank.

IV

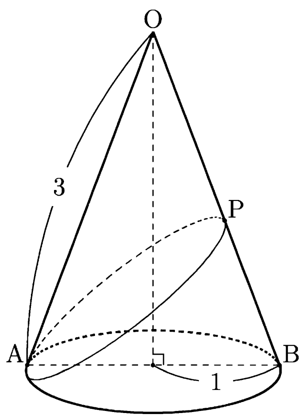
Let O be the vertex (apex) of a right circular cone such that the radius of the base is 1 and the slant height is 3.

(1) Consider the net of the cone, which consists of a sector and a circle. (The net of a solid is a 2-dimensional shape that can be folded to form that solid.) The central angle of the sector is **ABC** °, and the area of the sector is **D** π.

(2) Take two points A and B on the circumference of the base such that the line segment AB is a diameter. Take a point P on the segment OB and consider a path on the side of this circular cone which starts from the point A, passes through the point P and returns to A. Denote the length of the path by ℓ.

(i) If OP = 2, then the smallest value of ℓ is **E** $\sqrt{\textbf{F}}$.

(ii) Let point P be any point on the line segment OB. When ℓ is minimized, then OP $= \dfrac{\textbf{G}}{\textbf{H}}$, and the value of ℓ is **I** $\sqrt{\textbf{J}}$.



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This is the end of the questions for Part IV.

Leave the answer spaces **K** ～ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

If you do not correctly black out the appropriate oval, your answers will not be graded.

I

Q 1 Suppose we have a quadratic function $y = ax^2 + bx + c$ in x which satisfies the following conditions【$*$】:

【$*$】 When $x = -1$, then $y = -8$ and when $x = 3$, then $y = 16$. Further, in the interval $-1 \leqq x \leqq 3$, the value of y increases with the increase of the value of x.

We are to find the conditions which a, b and c must satisfy.

From【$*$】, it follows that b and c can be expressed in terms of a as

$$b = \boxed{\mathsf{AB}}\,a + \boxed{\mathsf{C}} \qquad \cdots\cdots\cdots \text{①}$$

$$c = \boxed{\mathsf{DE}}\,a - \boxed{\mathsf{F}}. \qquad \cdots\cdots\cdots \text{②}$$

Hence, the axis of symmetry of the graph of this quadratic function has the equation

$$x = \boxed{\mathsf{G}} - \frac{\boxed{\mathsf{H}}}{a}.$$

Thus a, b and c must satisfy the relationships ① and ②, and furthermore

$$0 < a \leqq \frac{\boxed{\mathsf{I}}}{\boxed{\mathsf{J}}} \quad \text{or} \quad \frac{\boxed{\mathsf{KL}}}{\boxed{\mathsf{M}}} \leqq a < 0.$$

- memo -

Q 2 Let a, b, c and d be real numbers satisfying $a < b < c < d$. Suppose that the two subsets of real numbers

$$A = \{x \mid a \leqq x \leqq c\}, \quad B = \{x \mid b \leqq x \leqq d\}$$

satisfy

$$A \cap B = \{x \mid x^2 - 4x + 3 \leqq 0\}.$$

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(1) Let the union of A and B be

$$A \cup B = \{x \mid x^2 - 5x - 24 \leqq 0\}.$$

Then the values of a, b, c and d are

$a =$ **NO**, $b =$ **P**, $c =$ **Q**, $d =$ **R**.

(2) Let the intersection of A and the complement $\overline{B}$ of B be

$$A \cap \overline{B} = \{x \mid x^2 + 5x - 6 \leqq 0 \text{ and } x \neq 1\},$$

and let the intersection of the complement $\overline{A}$ of A and B be

$$\overline{A} \cap B = \{x \mid x^2 - 9x + 18 \leqq 0 \text{ and } x \neq 3\}.$$

Then the values of a, b, c and d are

$a =$ **ST**, $b =$ **U**, $c =$ **V**, $d =$ **W**.

- memo -

This is the end of the questions for Part I. Leave the answer spaces **X** ～ **Z** of Part I blank.

II

Given a sphere S whose center is at O and whose radius is 1, take three points A, B and C on S such that

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \overrightarrow{OB} \cdot \overrightarrow{OC} = \overrightarrow{OC} \cdot \overrightarrow{OA} = 0.$$

Note that $\overrightarrow{OA} \cdot \overrightarrow{OB}$, etc., refers to the inner product of the two vectors.

(1) It follows that $\overrightarrow{AB} \cdot \overrightarrow{AC} = \boxed{\textbf{A}}$, $|\overrightarrow{AB}| = \sqrt{\boxed{\textbf{B}}}$, $\cos\angle BAC = \dfrac{\boxed{\textbf{C}}}{\boxed{\textbf{D}}}$

and the area of the triangle ABC is $\dfrac{\sqrt{\boxed{\textbf{E}}}}{\boxed{\textbf{F}}}$.

(2) Let G be the center of gravity of triangle ABC and P be the intersection point of the ray (half line) OG and sphere S.

Since $\overrightarrow{OG} = \dfrac{\boxed{\textbf{G}}}{\boxed{\textbf{H}}}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$, we have

$$|\overrightarrow{OG}| = \frac{\sqrt{\boxed{\textbf{I}}}}{\boxed{\textbf{J}}}, \quad |\overrightarrow{PG}| = \frac{\boxed{\textbf{K}} - \sqrt{\boxed{\textbf{L}}}}{\boxed{\textbf{M}}},$$

$$\overrightarrow{AG} \cdot \overrightarrow{PG} = \boxed{\textbf{N}}.$$

Hence the volume of the tetrahedron PABC is $\dfrac{\sqrt{\boxed{\textbf{O}}} - \boxed{\textbf{P}}}{\boxed{\textbf{Q}}}$.

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This is the end of the questions for Part II. Leave the answer spaces **R** ～ **Z** of Part II blank.

III

Given real numbers x and y that satisfy

$$\frac{x^2}{2} + \frac{y^2}{4} = 1, \quad x \geqq 0, \quad y \geqq 0,$$

we are to find the maximum value of

$$P = x^2 + xy + y^2.$$

Let x and y satisfy the conditions. When we set $x = \sqrt{2}\cos\theta \left(0 \leqq \theta \leqq \frac{\pi}{2}\right)$, we have

$$y = \boxed{\text{A}} \sin\theta.$$

Thus P can be represented as

$$\begin{aligned} P &= \sqrt{\boxed{\text{B}}}\sin 2\theta - \cos 2\theta + \boxed{\text{C}} \\ &= \sqrt{\boxed{\text{D}}}\sin(2\theta - \alpha) + \boxed{\text{E}}, \end{aligned}$$

where

$$\sin\alpha = \frac{\sqrt{\boxed{\text{F}}}}{\boxed{\text{G}}}, \quad \cos\alpha = \frac{\sqrt{\boxed{\text{H}}}}{\boxed{\text{I}}} \quad \left(0 < \alpha < \frac{\pi}{2}\right).$$

Hence the maximum value of P is $\sqrt{\boxed{\text{J}}} + \boxed{\text{K}}$.

Let us denote the θ at which the value of P is maximized by θ_0. Then we have

$$2\theta_0 = \alpha + \frac{\pi}{\boxed{\text{L}}},$$

and hence

$$\sin 2\theta_0 = \frac{\sqrt{\boxed{\text{M}}}}{\boxed{\text{N}}}, \quad \cos 2\theta_0 = -\frac{\sqrt{\boxed{\text{O}}}}{\boxed{\text{P}}}.$$

- memo -

This is the end of the questions for Part III. Leave the answer spaces **Q** ～ **Z** of Part III blank.

IV

Q 1 Let us define a sequence $\{S_n\}$ as

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \quad (n = 1, 2, 3, \cdots).$$

We are to find the following two limits:

$$\lim_{n\to\infty} S_n,$$

$$\lim_{n\to\infty} \frac{S_{2n} - S_n}{\sqrt{n}}.$$

(1) For each of **A** ～ **I** in the following sentences, choose the appropriate answer from among ⓪ ～ ⑨ at the bottom of this page.

Let us find $\lim_{n\to\infty} S_n$. Look at the function $y = \dfrac{1}{\sqrt{x}}$. We have

$$y' = -\frac{\boxed{\text{A}}}{2\sqrt{x^{\boxed{\text{B}}}}},$$

and hence this function y is **C**. So, considering each interval $k \leqq x \leqq k+1$ $(k = 1, 2, \cdots, n)$, we obtain

$$\frac{1}{\sqrt{k}} \;\boxed{\text{D}}\; \int_k^{k+1} \frac{1}{\sqrt{x}}\,dx.$$

When we separately add the left-hand sides and the right-hand sides of this expression from $k = 1$ to $k = n$, we have

$$S_n \;\boxed{\text{E}}\; \int_{\boxed{\text{F}}}^{\boxed{\text{G}}} \frac{1}{\sqrt{x}}\,dx = \boxed{\text{H}}\left(\sqrt{\boxed{\text{G}}} - 1\right)$$

and finally

$$\lim_{n\to\infty} S_n = \boxed{\text{I}}.$$

⓪	∞	①	1	②	2	③	3
④	n	⑤	$n+1$	⑥	$<$	⑦	$>$
⑧	monotonically increasing			⑨	monotonically decreasing		

(This question is continued on the next page.)

(2) For each of **J** ～ **P** in the following, choose the appropriate answer from among ⓪ ～ ⑨ below.

Let us find $\lim_{n\to\infty} \dfrac{S_{2n} - S_n}{\sqrt{n}}$. Since

$$S_{2n} - S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{\boxed{\mathsf{J}}}},$$

we have from quadrature (mensuration) by parts that

$$\lim_{n\to\infty} \frac{S_{2n} - S_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\boxed{\mathsf{K}}} \sum_{k=1}^{n} \frac{1}{\sqrt{\boxed{\mathsf{L}} + \dfrac{k}{n}}}$$

$$= \int_{\boxed{\mathsf{M}}}^{\boxed{\mathsf{N}}} \frac{1}{\sqrt{1+x}}\,dx$$

$$= \boxed{\mathsf{O}}\left(\sqrt{\boxed{\mathsf{P}}} - 1\right).$$

⓪ 0	① 1	② 2	③ $n-1$	④ n
⑤ $n+1$	⑥ $n-k$	⑦ $n+k$	⑧ $n+k-1$	⑨ $n+k+1$

Q 2 For each of **Q**, **S**, **V** in the following sentences, choose the appropriate expression from among ⓪ ～ ⑦ at the bottom of this page. For the other ☐, enter the correct number.

Suppose we have a differentiable function $f(x)$ which satisfies the equation

$$\int_0^x f(t)dt = (1+e^{-x})f(x) + 2x - 4\log 2. \quad \cdots\cdots\cdots \quad ①$$

We are to find $f(x)$ and the value of $\lim_{x\to\infty} f(x)$.

When we differentiate each side of ① with respect to x and transform the equation, we have

$$(1+e^{-x})\left(\boxed{\text{Q}}\right) = \boxed{\text{R}}. \quad \cdots\cdots\cdots \quad ②$$

Next we set $f(x) = e^x g(x)$, and using ②, we obtain

$$g'(x) = \frac{\boxed{\text{S}}}{1+e^{-x}}$$

and hence

$$g(x) = \boxed{\text{T}}\log(1+e^{-x}) + C,$$

where C is an integral constant.

Furthermore, since $g(0) = f(0)$, we see that $C = \boxed{\text{U}}$. Thus we obtain $g(x)$ and from that,

$$f(x) = \boxed{\text{V}}\log(1+e^{-x}).$$

Finally, we set $e^{-x} = t$ and obtain

$$f(x) = \boxed{\text{W}}\log(1+t)^{\frac{1}{t}}$$

and hence

$$\lim_{x\to\infty} f(x) = \lim_{t\to\boxed{\text{X}}} \boxed{\text{W}}\log(1+t)^{\frac{1}{t}} = \boxed{\text{Y}}.$$

⓪ $f'(x) - f(x)$	① $f(x) - f'(x)$	② $f'(x) - 2f(x)$
③ $f(x) - 2f'(x)$	④ $2e^x$	⑤ $-2e^x$
⑥ $2e^{-x}$	⑦ $-2e^{-x}$	

- memo -

This is the end of the questions for Part IV. Leave the answer space **Z** of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.