Question Booklet

2014 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, $\cdots$ in the questions represents a numeral (from 0 to 9) or the minus sign ($-$). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as [A] or [BC] is used later in the question. In such a case, the symbol is shaded when it is used later, as [A] or [BC].

Note the following:

(1) Reduce square roots ($\sqrt{\ }$) as much as possible.
(Example : Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

(2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3) If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4) If the answer to $\boxed{DE}\,x$ is $-x$, mark "$-$" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

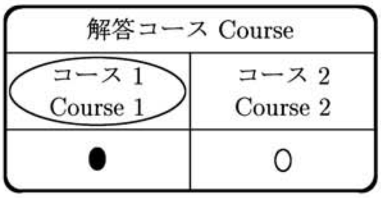
（Course 2 begins on page 15）

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.

< Example >



<u>**If you do not correctly fill in the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Let a and b be real numbers, where $a > 0$. Consider the two quadratic functions

$$f(x) = 2x^2 - 4x + 5, \quad g(x) = x^2 + ax + b.$$

We are to find the values of a and b when the function $g(x)$ satisfies the following two conditions.

(i) The minimum value of $g(x)$ is 8 less than the minimum value of $f(x)$.

(ii) There exists only one x which satisfies $f(x) = g(x)$.

Since the minimum value of $f(x)$ is $\boxed{\text{A}}$, from condition (i), we derive the equality

$$b = \frac{a^2}{\boxed{\text{B}}} - \boxed{\text{C}}.$$

Hence the equation from which we can find the x satisfying $f(x) = g(x)$ is

$$x^2 - (a + \boxed{\text{D}})x - \frac{a^2}{\boxed{\text{E}}} + \boxed{\text{FG}} = 0.$$

Thus, since $a > 0$, from condition (ii) we obtain

$$a = \boxed{\text{H}}, \quad b = \boxed{\text{IJ}}.$$

In this case, the x satisfying $f(x) = g(x)$ is $\boxed{\text{K}}$.

- memo -

Q 2 Consider the sets $A = \{4m \mid m \text{ is a natural number}\}$ and $B = \{6m \mid m \text{ is a natural number}\}$.

(1) For each of the following **L** ∼ **O**, choose the correct answer from among ⓪ ∼ ③ below.

Let n be a natural number.

(i) $n \in A$ is **L** for n to be divisible by 2.

(ii) $n \in B$ is **M** for n to be divisible by 24.

(iii) $n \in A \cup B$ is **N** for n to be divisible by 3.

(iv) $n \in A \cap B$ is **O** for n to be divisible by 12.

⓪ a necessary and sufficient condition
① a necessary condition but not a sufficient condition
② a sufficient condition but not a necessary condition
③ neither a necessary condition nor a sufficient condition

(2) Let $C = \{m \mid m \text{ is a natural number satisfying } 1 \leqq m \leqq 100\}$.

The number of elements which belong to $(\overline{A} \cup \overline{B}) \cap C$ is **PQ**, and the number of elements which belong to $\overline{A} \cap \overline{B} \cap C$ is **RS**. Note that $\overline{A}$ and $\overline{B}$ denote the complements of A and B, where the universal set is the set of all natural numbers.

- memo -

This is the end of the questions for Part I. Leave the answer spaces T ~ Z of Part I blank.

II

Q 1 Consider the permutations of the eight letters of the word “POSITION”.

(1) The number of permutations in which the two I’s are adjacent and the two O’s also are adjacent is **ABC**.

(2) The number of permutations such that the permutations both begin and end with the letter I and furthermore the two O’s are adjacent is **DEF**.

(3) The number of permutations that both begin and end with the letter I is **GHI**.

(4) The number of permutations of the 4 letters I, I, O, O is **J**. Also, the number of permutations of the 4 letters N, P, S, T is **KL**.

Hence the number of permutations of POSITION which begin or end with either I or O, and furthermore in which none of letters N, P, S, T are adjacent to each other is **MNO**.

- memo -

Q 2 Suppose that an integer x and a real number y satisfy both the equation

$$2(y+1) = x(8-x) \qquad \cdots\cdots\cdots \quad ①$$

and the inequality

$$5x - 4y + 1 \leqq 0. \qquad \cdots\cdots\cdots \quad ②$$

We are to find M, the maximum value of y, and m, the minimum value of y.

First of all, let us transform ① into

$$y = -\frac{1}{\boxed{\text{P}}}\left(x - \boxed{\text{Q}}\right)^2 + \boxed{\text{R}}.$$

Also, from ① and ② we obtain the inequality in x

$$2x^2 - \boxed{\text{ST}}\,x + \boxed{\text{U}} \leqq 0. \qquad \cdots\cdots\cdots \quad ③$$

Thus when x is an integer satisfying ③ if we consider the range of values which y can take, we see that y is maximized at $x = \boxed{\text{V}}$ and is minimized at $x = \boxed{\text{W}}$, and hence that

$$M = \boxed{\text{X}}, \quad m = \frac{\boxed{\text{Y}}}{\boxed{\text{Z}}}.$$

- memo -

This is the end of the questions for Part [II].

III

For each of **A** ～ **D** in the following questions, choose the correct answer from among ⓪ ～ ⑤ below each question.

Consider the three quadratic inequalities

$$x^2 + 3x - 18 < 0 \quad \cdots\cdots\cdots \quad ①$$

$$x^2 - 2x - 8 > 0 \quad \cdots\cdots\cdots \quad ②$$

$$x^2 + ax + b < 0. \quad \cdots\cdots\cdots \quad ③$$

(1) The range of x which satisfies both of the inequalities ① and ② is **A**.
Also, the range of x which satisfies neither inequality ① nor ② is **B**.

⓪ $3 \leqq x \leqq 4$　　① $-6 \leqq x \leqq -2$　　② $3 < x < 4$

③ $2 < x < 6$　　④ $-6 < x < -2$　　⑤ $-4 \leqq x \leqq -3$

(2) The range of x that satisfies at least one of the inequalities ① and ③ will be $-6 < x < 7$, if and only if a and b satisfy the equation **C**, and a satisfies the inequality **D**.

⓪ $b = 6a - 36$　　① $b = 7a - 49$　　② $b = -7a - 49$

③ $-10 < a \leqq -3$　　④ $-10 < a \leqq -1$　　⑤ $-1 \leqq a < 3$

- memo -

This is the end of the questions for Part [III]. Leave the answer spaces [E] ～ [Z] of Part [III] blank.

IV

Suppose that a quadrangle ABCD which is inscribed in a circle has the side lengths

$$AB = \sqrt{2}, \quad BC = CD = 2, \quad DA = \sqrt{6}.$$

(1) Let us set $\theta = \angle BAD$. We have the two equalities

$$BD^2 = \boxed{\mathsf{A}} - \boxed{\mathsf{B}}\sqrt{\boxed{\mathsf{C}}}\cos\theta,$$

$$BD^2 = \boxed{\mathsf{D}} + \boxed{\mathsf{E}}\cos\theta.$$

Hence,

$$\theta = \boxed{\mathsf{FG}}^\circ, \quad BD = \boxed{\mathsf{H}}\sqrt{\boxed{\mathsf{I}}}.$$

(2) Furthermore, we have

$$\angle BAC = \boxed{\mathsf{JK}}^\circ, \quad \angle BCA = \boxed{\mathsf{LM}}^\circ \quad \text{and} \quad AC = \boxed{\mathsf{N}} + \sqrt{\boxed{\mathsf{O}}}.$$

We also have

$$\sin\angle ADC = \frac{\sqrt{\boxed{\mathsf{P}}}\left(\sqrt{\boxed{\mathsf{Q}}} + \boxed{\mathsf{R}}\right)}{\boxed{\mathsf{S}}}.$$

(3) Let us denote the point of intersection of the straight line AD and the straight line BC by E. We have $EB = \boxed{\mathsf{T}} + \boxed{\mathsf{U}}\sqrt{\boxed{\mathsf{V}}}$.

- memo -

This is the end of the questions for Part IV.

Leave the answer spaces **W** ~ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

Do not take this question booklet out of the room.

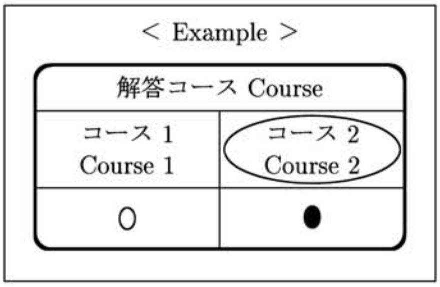
Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label “Course 2” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.



<u>**If you do not correctly fill in the appropriate oval, your answers will not be graded.**</u>

I

Q 1 Let a and b be real numbers, where $a > 0$. Consider the two quadratic functions

$$f(x) = 2x^2 - 4x + 5, \quad g(x) = x^2 + ax + b.$$

We are to find the values of a and b when the function $g(x)$ satisfies the following two conditions.

(i) The minimum value of $g(x)$ is 8 less than the minimum value of $f(x)$.

(ii) There exists only one x which satisfies $f(x) = g(x)$.

Since the minimum value of $f(x)$ is $\boxed{\text{A}}$, from condition (i), we derive the equality

$$b = \frac{a^2}{\boxed{\text{B}}} - \boxed{\text{C}}.$$

Hence the equation from which we can find the x satisfying $f(x) = g(x)$ is

$$x^2 - (a + \boxed{\text{D}})x - \frac{a^2}{\boxed{\text{E}}} + \boxed{\text{FG}} = 0.$$

Thus, since $a > 0$, from condition (ii) we obtain

$$a = \boxed{\text{H}}, \quad b = \boxed{\text{IJ}}.$$

In this case, the x satisfying $f(x) = g(x)$ is $\boxed{\text{K}}$.

- memo -

Q 2 Consider the sets $A = \{4m \mid m \text{ is a natural number}\}$ and $B = \{6m \mid m \text{ is a natural number}\}$.

(1) For each of the following **L** ～ **O**, choose the correct answer from among ⓪ ～ ③ below.

Let n be a natural number.

(i) $n \in A$ is **L** for n to be divisible by 2.

(ii) $n \in B$ is **M** for n to be divisible by 24.

(iii) $n \in A \cup B$ is **N** for n to be divisible by 3.

(iv) $n \in A \cap B$ is **O** for n to be divisible by 12.

⓪ a necessary and sufficient condition
① a necessary condition but not a sufficient condition
② a sufficient condition but not a necessary condition
③ neither a necessary condition nor a sufficient condition

(2) Let $C = \{m \mid m \text{ is a natural number satisfying } 1 \leqq m \leqq 100\}$.

The number of elements which belong to $(\overline{A} \cup \overline{B}) \cap C$ is **PQ**, and the number of elements which belong to $\overline{A} \cap \overline{B} \cap C$ is **RS**. Note that $\overline{A}$ and $\overline{B}$ denote the complements of A and B, where the universal set is the set of all natural numbers.

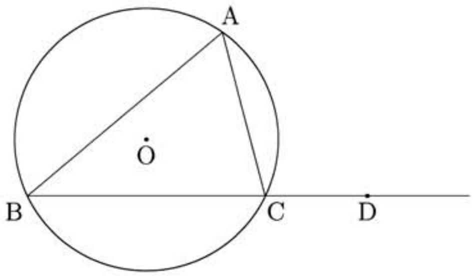
- memo -

This is the end of the questions for Part I. Leave the answer spaces **T** ～ **Z** of Part I blank.

II

Let S be a circle with its center at point O and a radius of 1. Let $\triangle$ABC be a triangle such that all its vertices are on S and AB : AC = 3 : 2. As shown in the figure, let D be a point on the extension of side BC and k be the number where

$$\text{BC} : \text{CD} = 2 : k.$$



Moreover, set

$$\overrightarrow{\text{OA}} = \vec{a}, \quad \overrightarrow{\text{OB}} = \vec{b}, \quad \overrightarrow{\text{OC}} = \vec{c}.$$

Answer the following questions.

(1) When we express $\overrightarrow{\text{OD}}$ in terms of $\vec{b}$, $\vec{c}$ and k, we have

$$\overrightarrow{\text{OD}} = \left(\frac{k}{\boxed{\text{A}}} + \boxed{\text{B}} \right) \vec{c} - \frac{k}{\boxed{\text{C}}} \vec{b}.$$

(2) Since the equality

$$\left| \vec{b} - \vec{a} \right| = \frac{\boxed{\text{D}}}{\boxed{\text{E}}} \left| \vec{c} - \vec{a} \right|$$

holds, by expressing the inner product $\vec{a} \cdot \vec{b}$ in terms of the inner product $\vec{a} \cdot \vec{c}$, we have

$$\vec{a} \cdot \vec{b} = \frac{\boxed{\text{F}}}{\boxed{\text{G}}} \vec{a} \cdot \vec{c} - \frac{\boxed{\text{H}}}{\boxed{\text{I}}}.$$

(3) It follows that when the tangent to S at the point A passes through the point D,

$$k = \frac{\boxed{\text{J}}}{\boxed{\text{K}}}.$$

- memo -

This is the end of the questions for Part [II]. Leave the answer spaces [L] ~ [Z] of Part [II] blank.

III

Let $p > 1$ and $q > 1$. Consider an equation in x

$$e^{2x} - ae^{x} + b = 0 \qquad \cdots\cdots\cdots \quad ①$$

such that the equation in t obtained by setting $t = e^{x}$ in ①

$$t^2 - at + b = 0$$

has the solutions $\log_{q^2} p$ and $\log_{p^3} q$.

We are to find the minimum value of a and the solution of equation ① at this minimum.

(1) First of all, we see that

$$b = \frac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}$$

and

$$a = \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}} \log_q p + \frac{\boxed{\mathsf{E}}}{\boxed{\mathsf{F}}} \log_p q.$$

(2) As long as $p > 1$ and $q > 1$, it always follows that $\log_p q > \boxed{\mathsf{G}}$. Hence, a takes the minimum value $\dfrac{\sqrt{\boxed{\mathsf{H}}}}{\boxed{\mathsf{I}}}$ when $\log_p q = \dfrac{\sqrt{\boxed{\mathsf{J}}}}{\boxed{\mathsf{K}}}$. In this case, the solution of ① is

$$x = -\frac{\boxed{\mathsf{L}}}{\boxed{\mathsf{M}}} \log_e \boxed{\mathsf{N}}.$$

- memo -

This is the end of the questions for Part [III]. Leave the answer spaces [O] ~ [Z] of Part [III] blank.

IV

Q 1 Let a and t be positive real numbers. Let ℓ be the tangent to the graph C of $y = ax^3$ at a point P (t, at^3), and let Q be the point at which ℓ intersects the curve C again. Further, let p be the line passing through the point P parallel to the x-axis; let q be the line passing through the point Q parallel to the y-axis; and let R be the point of intersection of p and q.

Also, let us denote by S_1 the area of the region bounded by the curve C, the straight line p and the straight line q, and denote by S_2 the area of the region bounded by the curve C and the tangent ℓ. We are to find the value of $\dfrac{S_1}{S_2}$.

First, since the equation of the tangent ℓ is

$$y = \boxed{\mathsf{A}}\, at^{\boxed{\mathsf{B}}}\, x - \boxed{\mathsf{C}}\, at^{\boxed{\mathsf{D}}},$$

the x-coordinate of Q is $-\boxed{\mathsf{E}}\, t$.

Hence, S_1 is

$$S_1 = \frac{\boxed{\mathsf{FG}}}{\boxed{\mathsf{H}}}\, at^{\boxed{\mathsf{I}}}.$$

Also, since S_2 is obtained by subtracting S_1 from the area of the triangle PQR, we have

$$S_2 = \frac{\boxed{\mathsf{JK}}}{\boxed{\mathsf{L}}}\, at^{\boxed{\mathsf{M}}}.$$

Hence, the value of $\dfrac{S_1}{S_2}$ is always

$$\frac{S_1}{S_2} = \boxed{\mathsf{N}},$$

independent of the values of a and t.

- memo -

Q 2 Given the function in x

$$f_n(x) = \sin^n x \quad (n = 1, 2, 3, \cdots),$$

answer the following questions.

(1) Consider the cases in which the equality

$$\lim_{x \to 0} \frac{a - x^2 - (b - x^2)^2}{f_n(x)} = c$$

holds for three real numbers a, b and c.

(i) We have $a = b^{\boxed{\mathrm{O}}}$.

(ii) When $n = 2$, if $c = 6$, then $b = \dfrac{\boxed{\mathrm{P}}}{\boxed{\mathrm{Q}}}$.

(iii) When $n = 4$, then $b = \dfrac{\boxed{\mathrm{R}}}{\boxed{\mathrm{S}}}$ and $c = -\boxed{\mathrm{T}}$.

(This question is continued on the next page.)

(2) For this $f_n(x)$, consider the definite integral

$$I_n = \int_0^{\frac{\pi}{2}} f_n(x) \sin 2x \, dx \quad (n = 1, 2, 3, \cdots).$$

When the integral is calculated, we have

$$I_n = \frac{\boxed{\mathsf{U}}}{n + \boxed{\mathsf{V}}}.$$

Hence we obtain

$$\lim_{n \to \infty} (I_{n-1} + I_n + I_{n+1} + \cdots + I_{2n-2}) = \int_0^{\boxed{\mathsf{W}}} \frac{\boxed{\mathsf{X}}}{\boxed{\mathsf{Y}} + x} \, dx$$

$$= \log \boxed{\mathsf{Z}}.$$

This is the end of the questions for Part $\boxed{\text{IV}}$.

This is the end of the questions for Course 2. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.