Question Booklet

2011 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instruction for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1−13, and Course 2 is on pages 15−27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instruction for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter A, B, C, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign(−). Completely fill in your answer for each letter in the corresponding line of the answer sheet(mark-sheet).

   Note the following :

   (1)Write square roots ($\sqrt{\ }$) in their simplest form.
   (Example : Substitute $2\sqrt{3}$ for $\sqrt{12}$.)

   (2)When writing a fraction, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
   (Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
   $-\frac{2}{\sqrt{6}}=\frac{-2\sqrt{6}}{6}=\frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

   (3)If your answer to $\frac{\boxed{A}\sqrt{\boxed{B}}}{\boxed{C}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

   (4)If the answer to $\boxed{DE}\,x$ is $-x$, mark "−" for D and "1" for E as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

3. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

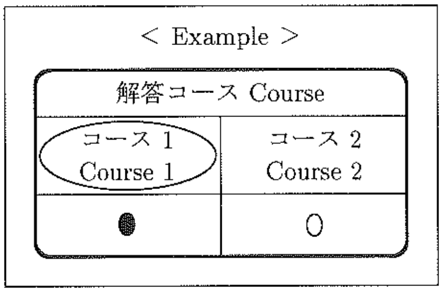
(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely black out the oval under the label on your answer sheet as shown in the example on the right.



If you do not correctly black out the appropriate oval, your answers will not be graded.

I

Q 1 Suppose that x and y satisfy

$$3x + y = 18, \quad x \geqq 1, \quad y \geqq 6.$$

We are to find the maximum value and the minimum value of xy.

When we express xy in terms of x, we have

$$xy = \boxed{\mathrm{AB}}\left(x - \boxed{\mathrm{C}}\right)^2 + \boxed{\mathrm{DE}}.$$

Also, the range of values which x can take is

$$\boxed{\mathrm{F}} \leqq x \leqq \boxed{\mathrm{G}}.$$

Hence, the value of xy is maximized at $x = \boxed{\mathrm{H}}$ and its value there is $\boxed{\mathrm{IJ}}$, and the value of xy is minimized at $x = \boxed{\mathrm{K}}$ and its value there is $\boxed{\mathrm{LM}}$.

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Q 2 Suppose that positive real numbers a and b satisfy

$$a^2 = 3 + \sqrt{5}, \quad b^2 = 3 - \sqrt{5}.$$

Let c be the fractional portion of $a + b$. We are to find the value of $\frac{1}{c} - c$.

(1) We see that $(ab)^2 = \boxed{\mathsf{N}}$ and $(a+b)^2 = \boxed{\mathsf{OP}}$.

(2) Since $\boxed{\mathsf{Q}} < a + b < \boxed{\mathsf{Q}} + 1$, the value of c is $\sqrt{\boxed{\mathsf{RS}}} - \boxed{\mathsf{T}}$.

Thus we obtain $\frac{1}{c} - c = \boxed{\mathsf{U}}$.

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This is the end of the questions for Part I. Leave the answer spaces V ~ Z of Part I blank.

II

Q 1 There are nine cards on which the integers from 1 to 9 are written in a box. Two cards are taken simultaneously from this box. Let S denote the sum of the numbers written on the two cards.

(1) The probability that S is 5 or less is $\frac{\boxed{\text{A}}}{\boxed{\text{B}}}$. Let us assign a score to the result S. When S is 5 or less the score is $10 - S$, and when it is greater than 5 the score is 2. Then the expected value of the score is $\frac{\boxed{\text{CD}}}{\boxed{\text{EF}}}$.

(2) Let us perform the above trial twice, returning the two cards to the box before the second trial.

(i) The probability that S is 5 or less in both trials is $\frac{\boxed{\text{G}}}{\boxed{\text{HI}}}$.

(ii) The probability that S is 5 or less in at least one trial is $\frac{\boxed{\text{JK}}}{\boxed{\text{LM}}}$.

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Q 2 Let a be a constant. For the two functions in x

$$f(x) = 2x^2 + x + a - 2$$
$$g(x) = -4x - 5,$$

we are to find the real values of x for which $f(x) = g(x)$ and also find the values of the two functions there.

(1) For each of $\boxed{\text{N}}$, $\boxed{\text{O}}$ and $\boxed{\text{P}}$ in the following statements, choose the appropriate condition from ⓪ ~ ⑧ below.

When $\boxed{\text{N}}$, there are two real values of x for which $f(x) = g(x)$.

When $\boxed{\text{O}}$, there is only one real value of x for which $f(x) = g(x)$.

When $\boxed{\text{P}}$, there is no real value of x for which $f(x) = g(x)$.

⓪ $a > \dfrac{1}{8}$ ① $a = \dfrac{17}{8}$ ② $a = \dfrac{1}{6}$ ③ $a < \dfrac{1}{6}$ ④ $a < \dfrac{17}{8}$

⑤ $a < \dfrac{1}{8}$ ⑥ $a > \dfrac{1}{6}$ ⑦ $a = \dfrac{1}{8}$ ⑧ $a > \dfrac{17}{8}$

(2) When $\boxed{\text{N}}$, the values of x for which $f(x) = g(x)$ are $\dfrac{-\boxed{\text{Q}} \pm \sqrt{\boxed{\text{R}} - \boxed{\text{S}}\,a}}{\boxed{\text{T}}}$,

and the values of the functions there are $\mp\sqrt{\boxed{\text{U}} - \boxed{\text{V}}\,a}$.

When $\boxed{\text{O}}$, the value of x for which $f(x) = g(x)$ is $-\dfrac{\boxed{\text{W}}}{\boxed{\text{X}}}$, and the value of the functions there is $\boxed{\text{Y}}$.

(3) Consider the case where $f(x) = g(x)$ and the absolute value of these functions there is greater than or equal to 3. The condition for this case is that $a \leqq -\boxed{\text{Z}}$.

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This is the end of the questions for Part [II].

III

Let a be a constant. Consider the quadratic function in x

$$y = 2x^2 + ax + 3. \qquad \cdots\cdots\cdots \quad ①$$

Suppose that the vertex of the graph of ① is in the first (upper right-hand) quadrant.

(1) The range of values which a can take is

$$\boxed{\mathsf{AB}}\sqrt{\boxed{\mathsf{C}}} < a < \boxed{\mathsf{D}},$$

and the least integer a satisfying this inequality is $\boxed{\mathsf{EF}}$.

(2) Let $a = \boxed{\mathsf{EF}}$ in ①. Let

$$y = 2x^2 + px + q$$

be the equation of the graph which is obtained by translating the graph of ① by $-\dfrac{1}{n}$ in the x-direction and by $\dfrac{6}{n^2}$ in the y-direction.

Then

$$p = \frac{\boxed{\mathsf{G}}}{n} - \boxed{\mathsf{H}}, \qquad q = \frac{\boxed{\mathsf{I}}}{n^2} - \frac{\boxed{\mathsf{J}}}{n} + \boxed{\mathsf{K}}.$$

(3) The total number of natural numbers n for which p in (2) is an integer is $\boxed{\mathsf{L}}$. Among these n, consider the ones such that the value of q is also an integer. Then $\boxed{\mathsf{M}}$ is the value of the n for which q takes the minimum value $\boxed{\mathsf{N}}$.

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This is the end of the questions for Part III. Leave the answer spaces O ~ Z of Part III blank.

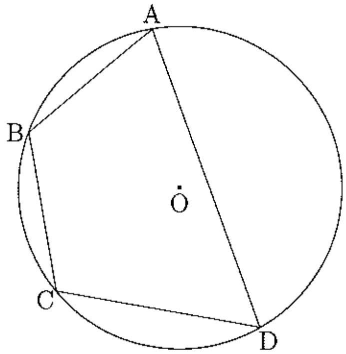
IV

A quadrangle ABCD which is inscribed in a circle O satisfies

$$\mathrm{AB} = \mathrm{BC} = \sqrt{2}, \quad \mathrm{BD} = \frac{3\sqrt{3}}{2}, \quad \angle\mathrm{ABC} = 120^\circ,$$

where

$$\mathrm{AD} > \mathrm{CD}. \quad \cdots\cdots\cdots \quad ①$$



(1) Then $\mathrm{AC} = \sqrt{\boxed{\mathbf{A}}}$, and the radius of circle O is $\sqrt{\boxed{\mathbf{B}}}$.

(2) Set $x = \mathrm{AD}$. Since $\angle\mathrm{ADB} = \boxed{\mathbf{CD}}^\circ$, x satisfies

$$4x^2 - \boxed{\mathbf{EF}}\,x + \boxed{\mathbf{GH}} = 0.$$

Also, set $y = \mathrm{CD}$. In the same way, it follows that y satisfies

$$4y^2 - \boxed{\mathbf{IJ}}\,y + \boxed{\mathbf{KL}} = 0.$$

Thus, noting ①, we obtain

$$\mathrm{AD} = \frac{\boxed{\mathbf{M}} + \sqrt{\boxed{\mathbf{N}}}}{\boxed{\mathbf{O}}}, \qquad \mathrm{CD} = \frac{\boxed{\mathbf{P}} - \sqrt{\boxed{\mathbf{Q}}}}{\boxed{\mathbf{R}}}.$$

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This is the end of the questions for Part IV.

Leave the answer spaces **S** ~ **Z** of Part IV blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

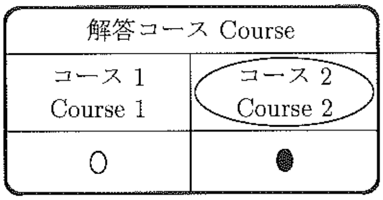
(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 2, for example, circle the label "Course 2" and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >



<u>If you do not correctly black out the appropriate oval, your answers will not be graded.</u>

I

Q 1 Suppose that x and y satisfy

$$3x + y = 18, \quad x \geqq 1, \quad y \geqq 6.$$

We are to find the maximum value and the minimum value of xy.

When we express xy in terms of x, we have

$$xy = \boxed{\mathsf{AB}}\left(x - \boxed{\mathsf{C}}\right)^2 + \boxed{\mathsf{DE}}.$$

Also, the range of values which x can take is

$$\boxed{\mathsf{F}} \leqq x \leqq \boxed{\mathsf{G}}.$$

Hence, the value of xy is maximized at $x = \boxed{\mathsf{H}}$ and its value there is $\boxed{\mathsf{IJ}}$, and the value of xy is minimized at $x = \boxed{\mathsf{K}}$ and its value there is $\boxed{\mathsf{LM}}$.

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Q 2 Suppose that positive real numbers a and b satisfy

$$a^2 = 3 + \sqrt{5}, \quad b^2 = 3 - \sqrt{5}.$$

Let c be the fractional portion of $a + b$. We are to find the value of $\dfrac{1}{c} - c$.

(1) We see that $(ab)^2 = \boxed{\mathsf{N}}$ and $(a+b)^2 = \boxed{\mathsf{OP}}$.

(2) Since $\boxed{\mathsf{Q}} < a + b < \boxed{\mathsf{Q}} + 1$, the value of c is $\sqrt{\boxed{\mathsf{RS}}} - \boxed{\mathsf{T}}$.

Thus we obtain $\dfrac{1}{c} - c = \boxed{\mathsf{U}}$.

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This is the end of the questions for Part [I]. Leave the answer spaces [V] ~ [Z] of Part [I] blank.

II

Given a sequence $\{a_n\}$ that satisfies the following conditions

$$a_1 = 1$$
$$a_{n+1} = 2{a_n}^2 \qquad (n = 1, 2, 3, \cdots), \qquad \cdots\cdots\cdots \quad ①$$

we are to find the number of natural numbers n satisfying $a_n < 10^{60}$. (For the value of $\log_{10} 2$, use the approximation 0.301.)

In this sequence we note that $a_n > 0$ for all natural numbers n. Thus when we consider common logarithms of both sides of ①, we have

$$\log_{10} a_{n+1} = \log_{10} \boxed{\mathbf{A}} + \boxed{\mathbf{B}} \log_{10} a_n .$$

When we set $b_n = \log_{10} a_n + \log_{10} \boxed{\mathrm{A}}$, the sequence $\{b_n\}$ is a geometric progression such that the common ratio is $\boxed{\mathbf{C}}$. Then

$$\log_{10} a_n = \left(\boxed{\mathbf{D}}^{\,n-1} - \boxed{\mathbf{E}} \right) \log_{10} \boxed{\mathbf{F}} .$$

Furthermore, since $a_n < 10^{60}$,

$$\boxed{\mathrm{D}}^{\,n-1} < \frac{\boxed{\mathbf{GH}}}{\log_{10} \boxed{\mathrm{F}}} + \boxed{\mathrm{E}} . \qquad \cdots\cdots\cdots \quad ②$$

Since $\boxed{\mathbf{IJK}}$ is the least natural number which is larger than the value of the right side of ②, the number of natural numbers n satisfying $a_n < 10^{60}$ is $\boxed{\mathbf{L}}$.

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This is the end of the questions for Part II. Leave the answer spaces M ~ Z of Part II blank.

III

Consider the following two equations

$$(\log_4 2\sqrt{x}\,)^2 + (\log_4 2\sqrt{y}\,)^2 = \log_2 (\sqrt[4]{2} \cdot x\sqrt{y}\,) \qquad \cdots\cdots\cdots \quad ①$$

$$\sqrt[3]{x} \cdot \sqrt[4]{y} = 2^k. \qquad \cdots\cdots\cdots \quad ②$$

We are to find the range of values which the constant k can take so that there exist positive real numbers x, y which satisfy ① and ② simultaneously.

Set $X = \log_2 x$ and $Y = \log_2 y$. Let us express ① and ② in terms of X and Y. First we consider ①. Since

$$\log_4 2\sqrt{x} = \frac{\log_2 x + \boxed{\mathsf{A}}}{\boxed{\mathsf{B}}}$$

and

$$\log_2 (\sqrt[4]{2} \cdot x\sqrt{y}\,) = \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}}} + \log_2 x + \frac{\log_2 y}{\boxed{\mathsf{E}}},$$

① reduces to

$$\left(X - \boxed{\mathsf{F}}\right)^2 + \left(Y - \boxed{\mathsf{G}}\right)^2 = \boxed{\mathsf{HI}}. \qquad \cdots\cdots\cdots \quad ③$$

In the same way, ② reduces to

$$4X + \boxed{\mathsf{J}}\,Y = \boxed{\mathsf{KL}}\,k. \qquad \cdots\cdots\cdots \quad ④$$

Since the distance d from the center of the circle ③ to the straight line ④ on the XY-plane is given by

$$d = \frac{\left|\boxed{\mathsf{MN}} - \boxed{\mathsf{OP}}\,k\right|}{\boxed{\mathsf{Q}}},$$

the range of values which k can take is

$$\boxed{\mathsf{R}} \leqq k \leqq \boxed{\mathsf{S}}.$$

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This is the end of the questions for Part III. Leave the answer spaces T ～ Z of Part III blank.

IV

Q 1 We are to differentiate

$$f(x) = \int_0^{2x} (t^2 - x^2) \sin 3t \, dt$$

with respect to x.

(1) We know that if $g(t)$ is a continuous function and $G(t)$ is one of its primitive functions, then

$$\int_0^{2x} g(t) \, dt = G(2x) - G(0) .$$

By differentiating both sides of this equality with respect to x, we have

$$\frac{d}{dx} \int_0^{2x} g(t) \, dt = \boxed{\mathbf{A}} ,$$

where $\boxed{\mathbf{A}}$ is the appropriate expression from among the following ⓪ ~ ⑦.

⓪ $g(x)$ ① $\frac{1}{2} g(x)$ ② $2g(x)$ ③ $g(2x)$

④ $\frac{1}{2} g(2x)$ ⑤ $2g(2x)$ ⑥ $g(x) - g(0)$ ⑦ $g(2x) - g(0)$

(2) We know that $f(x) = \int_0^{2x} t^2 \sin 3t \, dt - \int_0^{2x} x^2 \sin 3t \, dt$.

Since

$$\frac{d}{dx} \int_0^{2x} t^2 \sin 3t \, dt = \boxed{\mathbf{B}} x^2 \sin \boxed{\mathbf{C}} x$$

and

$$\frac{d}{dx} \int_0^{2x} x^2 \sin 3t \, dt = \frac{\boxed{\mathbf{D}}}{\boxed{\mathbf{E}}} x \left(-\cos \boxed{\mathbf{F}} x + \boxed{\mathbf{G}} + \boxed{\mathbf{H}} x \sin \boxed{\mathbf{I}} x \right) ,$$

we obtain

$$f'(x) = \frac{\boxed{\mathrm{D}}}{\boxed{\mathrm{E}}} x \left(\cos \boxed{\mathbf{J}} x - \boxed{\mathbf{K}} + \boxed{\mathbf{L}} x \sin \boxed{\mathbf{M}} x \right) .$$

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Q 2 Let a be a positive real number. Let P denote the point of intersection of the following two curves

$$C_1 : y = \frac{3}{x}$$
$$C_2 : y = \frac{a}{x^2},$$

and let ℓ denote the tangent to C_2 at P. Then we are to find the area S of the region bounded by C_1 and ℓ.

Since the coordinates of P are $\left(\frac{a}{\boxed{\text{N}}}, \frac{\boxed{\text{O}}}{a}\right)$, the equation of ℓ is

$$y = -\frac{\boxed{\text{PQ}}}{a^2}x + \frac{\boxed{\text{RS}}}{a}.$$

When we set

$$p = \frac{a}{\boxed{\text{T}}}, \qquad q = \frac{a}{\boxed{\text{U}}} \qquad (p < q),$$

S is obtained by calculating

$$S = \Big[\boxed{\text{V}}\Big]_p^q,$$

where $\boxed{\text{V}}$ is the appropriate expression from among ⓪ ~ ⑤ below.

Hence we obtain

$$S = \frac{\boxed{\text{W}}}{\boxed{\text{X}}} - 3\log\boxed{\text{Y}}.$$

⓪ $\frac{18}{a^2}x^2 - \frac{27}{a}x + 3\log|x|$

① $\frac{9}{a^2}x^2 - \frac{9}{a}x + 3\log|x|$

② $-\frac{27}{a^2}x^2 + \frac{18}{a}x - 3\log|x|$

③ $-\frac{27}{a^2}x^2 + \frac{27}{a}x - 3\log|x|$

④ $\frac{27}{a^2}x^2 - \frac{27}{a}x + 3\log|x|$

⑤ $-\frac{18}{a^2}x^2 + \frac{27}{a}x - 3\log|x|$

- memo -

This is the end of the questions for Part IV. Leave the answer space Z of Part IV blank.

This is the end of the questions for Course 2. Leave the answer spaces for Part V blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.