2021 Examination for Japanese University Admission for International Students

Mathematics (80 min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ Choose one of these courses and answer its questions only.

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instructions for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination voucher.
3. Course 1 is on pages 1-13, and Course 2 is on pages 15-27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instructions for how to answer the questions

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, ⋯ in the questions represents a numeral (from 0 to 9) or the minus sign (−). When you mark your answers, fill in the oval completely for each letter in the corresponding row of the answer sheet (mark-sheet).
3. Sometimes an answer such as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$ is used later in the question. In such a case, the symbol is shaded when it is used later, as $\boxed{\text{A}}$ or $\boxed{\text{BC}}$.

   Note the following :

   (1) Reduce square roots ($\sqrt{\ }$) as much as possible.
   (Example: Express $\sqrt{32}$ as $4\sqrt{2}$, not as $2\sqrt{8}$ or $\sqrt{32}$.)

   (2) For fractions, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
   (Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows:
   $-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

   (3) If your answer to $\frac{\boxed{\text{A}}\sqrt{\boxed{\text{B}}}}{\boxed{\text{C}}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

   (4) If the answer to $\boxed{\text{DE}}\,x$ is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

4. Carefully read the instructions on the answer sheet, too.

※ Once you are instructed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

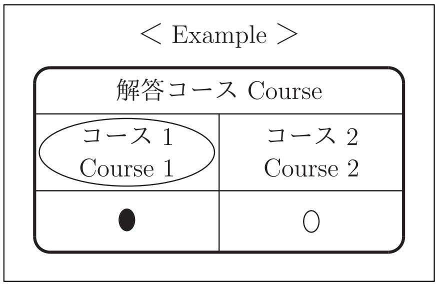
(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 1, for example, circle the label “Course 1” and completely fill in the oval under the label on your answer sheet as shown in the example on the right.



< Example >

解答コース Course	
コース 1 Course 1	コース 2 Course 2
●	○

If you do not correctly fill in the appropriate oval, your answers will not be graded.

$\boxed{\text{I}}$

Q 1　Consider the two quadratic functions

$$f(x) = -2x^2, \quad g(x) = x^2 + ax + b.$$

Function $g(x)$ satisfies the following two conditions:

(i)　the value of $g(x)$ is minimized at $x = 3$;

(ii)　$g(4) = f(4)$.

(1)　From condition (i) we see that $a = -\boxed{\textbf{A}}$. Further, from condition (ii) we see that $b = -\boxed{\textbf{BC}}$. Hence the minimum value of function $g(x)$ is $-\boxed{\textbf{DE}}$.

(2)　Let us find the value of x such that $f(x) = g(x)$ and x is not 4. Since x satisfies

$$x^2 - \boxed{\textbf{F}}\,x - \boxed{\textbf{G}} = 0,$$

we obtain $x = -\boxed{\textbf{H}}$.

(3)　The value of $f(x) - g(x)$ on $-\boxed{\text{H}} \leqq x \leqq 4$ is maximized at $x = \boxed{\textbf{I}}$, and its maximum value is $\boxed{\textbf{JK}}$.

- memo -

Q 2 For a game, each of two people, A and B, has a bag containing three cards on which the numbers 1, 2, and 3 are written, each number on a different card. In the game, A and B each take out one card from their own bag and compare the numbers. If the numbers are the same, the game is a draw. If the numbers are different, the person with the greater number wins.

(1) For a single game the probability of a draw is $\dfrac{\boxed{\mathbf{L}}}{\boxed{\mathbf{M}}}$.

(2) If this game is successively played four times, replacing the cards after each game, let us find the probabilities for the following.

(i) The probability that A wins three times or more is $\dfrac{\boxed{\mathbf{N}}}{\boxed{\mathbf{O}}}$.

(ii) The probability that A wins once and loses once and two games are draws is $\dfrac{\boxed{\mathbf{P}}}{\boxed{\mathbf{QR}}}$.

(iii) The probability that the number of times that A wins and the number of times that B wins are the same is $\dfrac{\boxed{\mathbf{ST}}}{\boxed{\mathbf{UV}}}$. Hence, the probability that the number of times that A wins is greater than the number of times that B wins is $\dfrac{\boxed{\mathbf{WX}}}{\boxed{\mathrm{UV}}}$.

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This is the end of the questions for $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{Y}}$, $\boxed{\textbf{Z}}$ of $\boxed{\text{I}}$ blank.

II

Q 1 Answer the following questions.

(1) The positive integers m and n which simultaneously satisfy the following two inequalities

$$\frac{m}{3} < \sqrt{3} < \frac{n}{4}, \quad \frac{n}{3} < \sqrt{6} < \frac{m}{2}$$

are

$$m = \boxed{\mathsf{A}}, \quad n = \boxed{\mathsf{B}}.$$

(2) Using the results of (1), let us compare the sizes of numbers ① ～ ⑤.

① $\left(\sqrt{(-3)(-4)}\right)^3$ ② $6\sqrt{(-2)^3(-3)}$ ③ $\sqrt{\{(-4)(-3)^2\}^2}$

④ $(-1)^3\sqrt{\{(-2)^5\}^2}$ ⑤ $\left(\frac{5\sqrt{3}}{1-\sqrt{6}}\right)^2$

When the denominator of ⑤ is rationalized, we have

$$\left(\frac{5\sqrt{3}}{1-\sqrt{6}}\right)^2 = \boxed{\mathsf{CD}} + \boxed{\mathsf{E}}\sqrt{\boxed{\mathsf{F}}}.$$

Of the five numbers, there are $\boxed{\mathsf{G}}$ number(s) greater than 35 and $\boxed{\mathsf{H}}$ negative number(s).

When we arrange the five numbers in the ascending order of their size using the numbers ① ～ ⑤, we have

$$\boxed{\mathsf{I}} < \boxed{\mathsf{J}} < \boxed{\mathsf{K}} < \boxed{\mathsf{L}} < \boxed{\mathsf{M}}.$$

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Q 2 The function $f(x) = x^2 + ax + b$ satisfies the following two conditions:

(i) $f(3) = 1$;

(ii) $13 \leqq f(-1) \leqq 25$.

We are to express the minimum value m of $f(x)$ in terms of a. In addition, we are to find the maximum and minimum values of m.

From condition (i), a and b satisfy

$$\boxed{\mathsf{N}}\,a + b + \boxed{\mathsf{O}} = 0.$$

From this, $f(x)$ can be expressed in terms of a as

$$f(x) = x^2 + ax - \boxed{\mathsf{P}}\,a - \boxed{\mathsf{Q}}.$$

Hence from condition (ii), a satisfies

$$-\boxed{\mathsf{R}} \leqq a \leqq -\boxed{\mathsf{S}}.$$

On the other hand, m can be expressed in terms of a as

$$m = -\frac{1}{\boxed{\mathsf{T}}}\left(a + \boxed{\mathsf{U}}\right)^2 + \boxed{\mathsf{V}}.$$

Thus m is maximized at $a = -\boxed{\mathsf{W}}$, and its maximum value is $\boxed{\mathsf{X}}$; it is minimized at $a = -\boxed{\mathsf{Y}}$, and its minimum value is $\boxed{\mathsf{Z}}$.

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This is the end of the questions for $\boxed{\text{II}}$.

III

Let N be a positive integer. Both when it is written in base 5 and when it is written in base 9, it is a 3-digit number, but the order of the numerals is reversed. We are to represent N in base 10 (decimal) and in base 4.

Let N be abc in base 5 and cba in base 9. Then we have

$$\boxed{\mathbf{A}} \leqq a \leqq \boxed{\mathbf{B}}, \quad \boxed{\mathbf{C}} \leqq b \leqq \boxed{\mathbf{D}}, \quad \boxed{\mathbf{E}} \leqq c \leqq \boxed{\mathbf{F}}. \quad \cdots\cdots\cdots ①$$

Since we also have

$$N = \boxed{\mathbf{GH}}\,a + \boxed{\mathbf{I}}\,b + c = \boxed{\mathbf{JK}}\,c + \boxed{\mathbf{L}}\,b + a,$$

we obtain

$$b = \boxed{\mathbf{M}}\,a - \boxed{\mathbf{NO}}\,c. \quad \cdots\cdots\cdots ②$$

The a, b and c satisfying ① and ② are

$$a = \boxed{\mathbf{P}}, \quad b = \boxed{\mathbf{Q}}, \quad c = \boxed{\mathbf{R}}.$$

Thus N expressed in base 10 is $\boxed{\mathbf{STU}}$, and N expressed in base 4 is $\boxed{\mathbf{VWXY}}$.

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This is the end of the questions for $\boxed{\text{III}}$. Leave the answer space $\boxed{\textsf{Z}}$ of $\boxed{\text{III}}$ blank.

IV

In a triangle ABC, let ∠B = 45° and ∠C = 75°, and let D be the intersection of the bisector of ∠A and side BC.

(1) From the law of sines we have

$$\mathrm{AC} = \frac{\sqrt{\boxed{\mathbf{A}}}}{\boxed{\mathbf{B}}}\,\mathrm{BC}, \qquad \mathrm{AD} = \sqrt{\boxed{\mathbf{C}}}\,\mathrm{BD}.$$

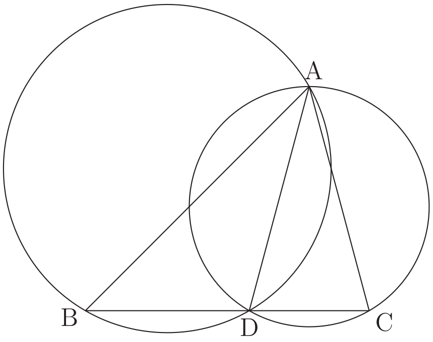
In particular, from $\angle\mathrm{ADC} = \boxed{\mathbf{DE}}^{\circ}$ we see that

$$\mathrm{BD} : \mathrm{BC} = \boxed{\mathbf{F}} : \sqrt{\boxed{\mathbf{G}}}$$

and hence we have

$$\mathrm{AB} : \mathrm{AC} = \boxed{\mathbf{H}} : \left(\sqrt{\boxed{\mathbf{I}}} - \boxed{\mathbf{J}}\right).$$

(2) Let O_1 be the center of the circumscribed circle of triangle ABD, and let O_2 be the center of the circumscribed circle of triangle ADC. Let us find the ratio of the areas of triangle ABC and triangle $\mathrm{AO}_1\mathrm{O}_2$, $\triangle\mathrm{ABC} : \triangle\mathrm{AO}_1\mathrm{O}_2$.



Since $\angle\mathrm{AO}_1\mathrm{D} = \boxed{\mathbf{KL}}^{\circ}$ and $\angle\mathrm{AO}_2\mathrm{O}_1 = \boxed{\mathbf{MN}}^{\circ}$, by the same reasoning as (1), we have

$$\mathrm{AC} = \sqrt{\boxed{\mathbf{O}}}\,\mathrm{AO}_1, \qquad \mathrm{AO}_2 = \left(\sqrt{\boxed{\mathbf{P}}} - \boxed{\mathbf{Q}}\right)\mathrm{AO}_1.$$

Hence we obtain

$$\triangle\mathrm{ABC} : \triangle\mathrm{AO}_1\mathrm{O}_2 = \boxed{\mathbf{R}} : \left(\boxed{\mathbf{S}} - \sqrt{\boxed{\mathbf{T}}}\right).$$

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This is the end of the questions for $\boxed{\text{IV}}$. Leave the answer spaces $\boxed{\textbf{U}}$ ～ $\boxed{\textbf{Z}}$ of $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 1. Leave the answer spaces for $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

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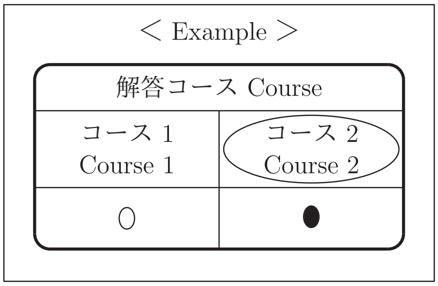

＜ Example ＞

解答コース Course	
コース 1 Course 1	[✓ コース 2 Course 2]
○	●

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- memo -

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$\boxed{\text{II}}$

Q 1 For $\boxed{\textbf{C}}$, $\boxed{\textbf{D}}$, $\boxed{\textbf{E}}$, $\boxed{\textbf{F}}$, $\boxed{\textbf{G}}$ in the following sentences, choose the correct answer from among choices ⓪ ～ ⑨ below. For the other $\boxed{}$, enter the correct number.

Consider a regular tetrahedron OABC with sides of length 1. Let x be a number satisfying $0 < x < 1$, and let P be the point that divides side AB by the ratio $x : (1-x)$ and Q be the point that divides side BC by the ratio $x : (1-x)$. Also, let $\overrightarrow{\mathrm{OA}} = \vec{a}$, $\overrightarrow{\mathrm{OB}} = \vec{b}$ and $\overrightarrow{\mathrm{OC}} = \vec{c}$. We are to find the range of values of $\cos\angle\mathrm{POQ}$.

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = \frac{\boxed{\textbf{A}}}{\boxed{\textbf{B}}}.$$

Next, since we can express $\overrightarrow{\mathrm{OP}}$ and $\overrightarrow{\mathrm{OQ}}$ as $\overrightarrow{\mathrm{OP}} = \boxed{\textbf{C}}$ and $\overrightarrow{\mathrm{OQ}} = \boxed{\textbf{D}}$, we have

$$\left|\overrightarrow{\mathrm{OP}}\right| = \left|\overrightarrow{\mathrm{OQ}}\right| = \sqrt{\boxed{\textbf{E}}}, \qquad \overrightarrow{\mathrm{OP}}\cdot\overrightarrow{\mathrm{OQ}} = \boxed{\textbf{F}}.$$

Hence we obtain

$$\cos\angle\mathrm{POQ} = \frac{1}{\boxed{\textbf{G}}} - \frac{\boxed{\textbf{H}}}{\boxed{\textbf{I}}}.$$

From this we finally obtain

$$\frac{\boxed{\textbf{J}}}{\boxed{\textbf{K}}} < \cos\angle\mathrm{POQ} \leqq \frac{\boxed{\textbf{L}}}{\boxed{\textbf{M}}}.$$

⓪ $(1-x)\vec{a} + x\vec{b}$

① $x\vec{a} + (1-x)\vec{b}$

② $(1-x)\vec{b} + x\vec{c}$

③ $x\vec{b} + (1-x)\vec{c}$

④ $x^2 + x + 1$

⑤ $x^2 - x + 1$

⑥ $x^2 - x - 1$

⑦ $\frac{1}{2}\left(-x^2 + x + 1\right)$

⑧ $\frac{1}{2}\left(-x^2 - x + 1\right)$

⑨ $\frac{1}{2}\left(-x^2 + x - 1\right)$

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Q 2 We have a triangle ABC on the complex plane whose vertices are the three points A(α), B(β) and C(γ) that satisfy

$$\frac{\gamma-\alpha}{\beta-\alpha}=1-i.$$

(In the following, the range of an argument θ is $0 \leqq \theta < 2\pi$.)

(1) When we express the complex number $\frac{\gamma-\alpha}{\beta-\alpha}$ in polar form, we have

$$\frac{\gamma-\alpha}{\beta-\alpha}=\sqrt{\boxed{\textbf{N}}}\left(\cos\frac{\boxed{\textbf{O}}}{\boxed{\textbf{P}}}\pi+i\sin\frac{\boxed{\text{O}}}{\boxed{\text{P}}}\pi\right).$$

Hence we see that point C is the point resulting from rotating point B by $\frac{\boxed{\textbf{Q}}}{\boxed{\textbf{R}}}\pi$ around point A and then changing its distance from point A to its distance multiplied by $\sqrt{\boxed{\textbf{S}}}$. From this we also see that the absolute value and the argument of the complex number $w=\frac{\gamma-\beta}{\alpha-\beta}$ are

$$|w|=\boxed{\textbf{T}} \quad \text{and} \quad \arg w=\frac{\boxed{\textbf{U}}}{\boxed{\textbf{V}}}\pi.$$

(2) If $\alpha+\beta+\gamma=0$, then we have that

$$|\alpha|:|\beta|:|\gamma|=\sqrt{\boxed{\textbf{W}}}:\sqrt{\boxed{\textbf{X}}}:\sqrt{\boxed{\textbf{Y}}}.$$

- memo -

This is the end of the questions for $\boxed{\text{II}}$. Leave the answer space $\boxed{\textbf{Z}}$ of $\boxed{\text{II}}$ blank.

III

We are to find the minimum value of the function

$$f(x) = 8^x + 8^{-x} - 3\left(4^{1+x} + 4^{1-x} - 2^{4+x} - 2^{4-x}\right) - 24$$

and the value of x at which the function takes this minimum value.

First, let us set $2^x + 2^{-x} = t$. Then, since

$$4^x + 4^{-x} = t^2 - \boxed{\mathbf{A}} \qquad \text{and} \qquad 8^x + 8^{-x} = t^3 - \boxed{\mathbf{B}}\,t,$$

$f(x)$ can be expressed as

$$f(x) = t^3 - \boxed{\mathbf{CD}}\,t^2 + \boxed{\mathbf{EF}}\,t.$$

When we consider the right side as a function of t and denote it by $g(t)$, its derivative is

$$g'(t) = \boxed{\mathbf{G}}\left(t - \boxed{\mathbf{H}}\right)\left(t - \boxed{\mathbf{I}}\right),$$

where $\boxed{\text{H}} < \boxed{\text{I}}$.

Here, since $2^x + 2^{-x} = t$, the range of the values which t takes is

$$t \geqq \boxed{\mathbf{J}}.$$

When $t = \boxed{\text{J}}$, we see that $g\left(\boxed{\text{J}}\right) = \boxed{\mathbf{KL}}$. When $t > \boxed{\text{J}}$, $g(t)$ is

locally maximized at $t = \boxed{\mathbf{M}}$, and its local maximum is $\boxed{\mathbf{NO}}$,

and furthermore, it is

locally minimized at $t = \boxed{\mathbf{P}}$, and its local minimum is $\boxed{\mathbf{QR}}$.

Thus, the minimum value of $f(x)$ is $\boxed{\mathbf{ST}}$, which is taken at

$$x = \boxed{\mathbf{U}} \qquad \text{and} \qquad x = \log_2\left(\boxed{\mathbf{V}} \pm \sqrt{\boxed{\mathbf{WX}}}\right) - \boxed{\mathbf{Y}}.$$

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This is the end of the questions for $\boxed{\text{III}}$. Leave the answer space $\boxed{\textbf{Z}}$ of $\boxed{\text{III}}$ blank.

IV

Let k be a positive real number. Consider the two curves

$$C_1 : y = \sin^2 x, \qquad C_2 : y = k\cos 2x \qquad \left(0 \leqq x \leqq \frac{\pi}{2}\right).$$

Let S_1 be the area of the region bounded by the two curves C_1, C_2 and the y-axis, and let S_2 be the area of the region bounded by the two curves C_1, C_2 and the straight line $x = \dfrac{\pi}{2}$. We are to show that the value of $S_2 - S_1$ is a constant independent of the value of k.

When we denote the x satisfying the equation $\sin^2 x = k\cos 2x$ by α, we have

$$\sin\alpha = \sqrt{\frac{k}{\boxed{\mathbf{A}}\,k + \boxed{\mathbf{B}}}}, \qquad \cos\alpha = \sqrt{\frac{k + \boxed{\mathbf{C}}}{\boxed{\mathbf{D}}\,k + \boxed{\mathbf{E}}}}.$$

Then we have

$$S_1 = \frac{\boxed{\mathbf{F}}}{\boxed{\mathbf{G}}} \int_0^{\alpha} \left\{\left(\boxed{\mathbf{H}}\,k + \boxed{\mathbf{I}}\right)\cos \boxed{\mathbf{J}}\,x - 1\right\} dx$$

$$= \frac{\boxed{\mathbf{K}}}{\boxed{\mathbf{L}}} \left\{\sqrt{k\left(k + \boxed{\mathbf{M}}\right)} - \alpha\right\},$$

$$S_2 = \frac{\boxed{\mathbf{N}}}{\boxed{\mathbf{O}}} \left\{\sqrt{k\left(k + \boxed{\mathbf{P}}\right)} - \alpha\right\} + \frac{\pi}{\boxed{\mathbf{Q}}}.$$

Hence, we obtain

$$S_2 - S_1 = \frac{\pi}{\boxed{\mathbf{R}}},$$

which shows that the value of $S_2 - S_1$ is a constant independent of the value of k.

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This is the end of the questions for $\boxed{\text{IV}}$. Leave the answer spaces $\boxed{\textbf{S}}$ ～ $\boxed{\textbf{Z}}$ of $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 2. Leave the answer spaces for $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.