Question Booklet

2010 Examination for Japanese University Admission for International Students

Mathematics (80min.)

【Course 1 (Basic), Course 2 (Advanced)】

※ **Choose one of these courses and answer its questions only.**

I Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

II Instruction for the Question Booklet

1. Do not open this question booklet until instructed.
2. After being instructed, write your name and examination registration number in space provided below, as printed on your examination admission card.
3. Course 1 is on pages 1−13, and Course 2 is on pages 15−27.
4. If your question booklet is missing any pages, raise your hand.
5. You may write notes and calculations in the question booklet.

III Instruction for the Answer Sheet

1. You must mark your answers on the answer sheet with an HB pencil.
2. Each letter **A**, **B**, **C**, $\cdots$ in the questions represents a numeral (from 0 to 9) or the minus sign (−). Completely fill in your answer for each letter in the corresponding line of the answer sheet (mark-sheet).

Note the following :

(1)Write square roots ($\sqrt{\ }$) in their simplest form.
(Example : Substitute $2\sqrt{3}$ for $\sqrt{12}$.)

(2)When writing a fraction, attach the minus sign to the numerator, and reduce the fraction to its lowest terms.
(Example : Substitute $\frac{1}{3}$ for $\frac{2}{6}$. Also simplify as follows :
$-\frac{2}{\sqrt{6}} = \frac{-2\sqrt{6}}{6} = \frac{-\sqrt{6}}{3}$. Then apply $\frac{-\sqrt{6}}{3}$ to the answer.)

(3)If your answer to $\frac{\boxed{\text{A}}\sqrt{\boxed{\text{B}}}}{\boxed{\text{C}}}$ is $\frac{-\sqrt{3}}{4}$, mark as shown below.

(4)If the answer to $\boxed{\text{DE}}\,x$ is $-x$, mark "−" for **D** and "1" for **E** as shown below.

A	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
B	⊖	⓪	①	②	●	④	⑤	⑥	⑦	⑧	⑨
C	⊖	⓪	①	②	③	●	⑤	⑥	⑦	⑧	⑨
D	●	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
E	⊖	⓪	●	②	③	④	⑤	⑥	⑦	⑧	⑨

3. Carefully read the instructions on the answer sheet, too.

※ Once you are informed to start the examination, fill in your examination registration number and name.

Examination registration number			*					*						
Name														

Mathematics Course 1

(Basic Course)

(Course 2 begins on page 15)

Marking Your Choice of Course on the Answer Sheet

Choose to answer <u>either</u> Course 1 or Course 2.

If you choose Course 1, for example, circle the label "Course 1" and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >

解答コース Course	
[✓ コース 1 Course 1]	コース 2 Course 2
●	○

<u>If you do not correctly black out the appropriate oval, your answers will not be graded.</u>

$\boxed{\text{I}}$

Q 1 Consider the equation

$$(x-1)^2 = |3x-5|. \quad \cdots\cdots\cdots \quad ①$$

(1) Among all solutions of equation ①, the solutions satisfying $x \geqq \frac{5}{3}$ are $x = \boxed{\textbf{A}}$ and $x = \boxed{\textbf{B}}$, where $\boxed{\text{A}} < \boxed{\text{B}}$.

(2) Equation ① has a total of $\boxed{\textbf{C}}$ solutions. When the minimum one is denoted by α, the integer m satisfying $m-1 < \alpha \leqq m$ is $\boxed{\textbf{DE}}$.

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Q 2 Consider the following three conditions (a), (b) and (c) on two real numbers x and y:

(a) $x+y=5$ and $xy=3$,

(b) $x+y=5$ and $x^2+y^2=19$,

(c) $x^2+y^2=19$ and $xy=3$.

(1) Using the equality $x^2+y^2=(x+y)^2-\boxed{\text{F}}\,xy$, we see that

condition (b) gives $xy=\boxed{\text{G}}$,

condition (c) gives $x+y=\boxed{\text{H}}$ or $x+y=\boxed{\text{IJ}}$.

(2) For each of the following $\boxed{\text{K}}\sim\boxed{\text{M}}$, choose the most appropriate answer from among the choices ⓪ ~ ③ below.

(i) (a) is $\boxed{\text{K}}$ for (b).

(ii) (b) is $\boxed{\text{L}}$ for (c).

(iii) (c) is $\boxed{\text{M}}$ for (a).

⓪ a necessary and sufficient condition
① a sufficient condition but not a necessary condition
② a necessary condition but not a sufficient condition
③ neither a necessary condition nor a sufficient condition

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This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{N}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

II

Q 1 Using the five numerals $0, 1, 2, 3, 4$, we are to make four-digit integers. (Note that "0123", etc. are not allowed.)

(1) The total possible number of integers where the digits are all different numerals is $\boxed{\mathsf{AB}}$. Among them, the total number of integers that do not use 0 is $\boxed{\mathsf{CD}}$.

(2) If we are allowed to use the same numeral repeatedly, then the total possible number of four-digit integers is $\boxed{\mathsf{EFG}}$. Among them

(i) the total number of integers that use both 1 and 3 twice is $\boxed{\mathsf{H}}$,

(ii) the total number of integers that use both 0 and 4 twice is $\boxed{\mathsf{I}}$,

(iii) the total number of integers that use both of two numerals twice is $\boxed{\mathsf{JK}}$.

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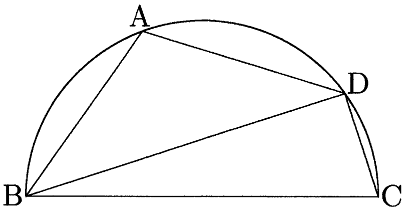
Q 2 As shown in the figure, a triangle ABD is inscribed in a semi-circle with the diameter BC, where

$$\mathrm{AB} = 3, \quad \mathrm{BD} = 5, \quad \tan\angle\mathrm{ABD} = \frac{3}{4}.$$



We are to find the lengths of the three sides BC, CD and DA of the quadrangle ABCD and the area S of the quadrangle ABCD.

First, since $\cos\angle\mathrm{ABD} = \dfrac{\boxed{\mathsf{L}}}{\boxed{\mathsf{M}}}$, we have $\mathrm{DA} = \sqrt{\boxed{\mathsf{NO}}}$.

Also, since $\sin\angle\mathrm{ABD} = \dfrac{\boxed{\mathsf{P}}}{\boxed{\mathsf{Q}}}$, we have $\mathrm{BC} = \dfrac{\boxed{\mathsf{R}}\sqrt{\boxed{\mathsf{ST}}}}{\boxed{\mathsf{U}}}$ and thus $\mathrm{CD} = \dfrac{\boxed{\mathsf{V}}}{\boxed{\mathsf{W}}}$. From these we obtain

$$S = \frac{\boxed{\mathsf{XY}}}{\boxed{\mathsf{Z}}}.$$

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This is the end of the questions for Part $\boxed{\text{II}}$.

III

Consider the following quadratic equations in x

$$x^2 + 2x - 15 = 0 \qquad \cdots\cdots\cdots \quad ①$$
$$2x^2 + 3x + a^2 + 12a = 0. \qquad \cdots\cdots\cdots \quad ②$$

Let us denote the two solutions of ① by α and β $(\alpha < \beta)$. We are to find the range of values which a in ② can take, in order that ② has two real solutions γ and δ and they satisfy

$$\alpha < \gamma < \beta < \delta.$$

(1) $\alpha = \boxed{\mathsf{AB}}$ and $\beta = \boxed{\mathsf{C}}$.

(2) When we set $b = a^2 + 12a$, from the condition $\alpha < \gamma$ we have

$$b > \boxed{\mathsf{DEF}},$$

and from the condition $\gamma < \beta < \delta$ we have

$$b < \boxed{\mathsf{GHI}}.$$

Hence the range of the values which a can take is

$$\boxed{\mathsf{JK}} < a < \boxed{\mathsf{LM}}, \quad \boxed{\mathsf{NO}} < a < \boxed{\mathsf{PQ}},$$

where $\boxed{\mathsf{JK}} < \boxed{\mathsf{NO}}$.

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This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\textbf{R}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{III}}$ blank.

IV

Let x, y and z satisfy the following two equations:

$$x + y - z = 0 \qquad \cdots\cdots\cdots \quad ①$$

$$2x - y + 1 = 0. \qquad \cdots\cdots\cdots \quad ②$$

We are to find the values of a, b and c such that the equation

$$ax^2 + by^2 + cz^2 = 1 \qquad \cdots\cdots\cdots \quad ③$$

holds for all x, y and z satisfying ① and ②.

First, given ① and ②, we may express y and z in terms of x as

$$y = \boxed{\mathsf{A}}\,x + \boxed{\mathsf{B}}, \quad z = \boxed{\mathsf{C}}\,x + \boxed{\mathsf{D}}. \qquad \cdots\cdots\cdots \quad ④$$

This shows that the values of both y and z depend on the value of x.

Next, when ④ is substituted into ③ and the left side is arranged in descending order of powers of x, we obtain

$$(a + \boxed{\mathsf{E}}\,b + \boxed{\mathsf{F}}\,c)x^2 + (\boxed{\mathsf{G}}\,b + \boxed{\mathsf{H}}\,c)x + b + c = 1.$$

Since this equation holds for any x, it holds also when $x = 0$, $x = 1$ and $x = -1$ are substituted into it, from which we obtain

$$\begin{cases} b + c = 1 \\ a + 9b + \boxed{\mathsf{IJ}}\,c = 1 \\ a + b + \boxed{\mathsf{K}}\,c = 1\,. \end{cases}$$

When we regard these as simultaneous equations and solve them for a, b and c, we have

$$a = \boxed{\mathsf{L}}, \qquad b = \boxed{\mathsf{M}}, \qquad c = \boxed{\mathsf{NO}}.$$

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This is the end of the questions for Part $\boxed{\text{IV}}$.

Leave the answer spaces $\boxed{\textbf{P}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{IV}}$ blank.

This is the end of the questions for Course 1. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 1" on your answer sheet.

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Mathematics Course 2

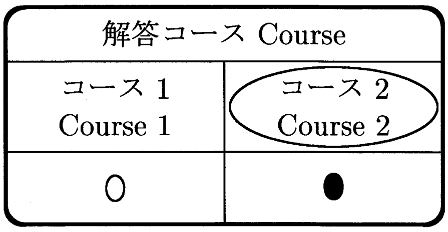
(Advanced Course)

Marking Your Choice of Course on the Answer Sheet

Choose to answer either Course 1 or Course 2.

If you choose Course 2, for example, circle the label "Course 2" and completely black out the oval under the label on your answer sheet as shown in the example on the right.

< Example >



解答コース Course	
コース 1 Course 1	コース 2 Course 2
○	●

If you do not correctly black out the appropriate oval, your answers will not be graded.

$\boxed{\text{I}}$

Q 1 Consider the equation

$$(x-1)^2 = |3x-5|. \qquad \cdots\cdots\cdots \quad ①$$

(1) Among all solutions of equation ①, the solutions satisfying $x \geqq \frac{5}{3}$ are $x = \boxed{\mathbf{A}}$ and $x = \boxed{\mathbf{B}}$, where $\boxed{\text{A}} < \boxed{\text{B}}$.

(2) Equation ① has a total of $\boxed{\mathbf{C}}$ solutions. When the minimum one is denoted by α, the integer m satisfying $m-1 < \alpha \leqq m$ is $\boxed{\mathbf{DE}}$.

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Q 2 Consider the following three conditions (a), (b) and (c) on two real numbers x and y:

(a) $x+y=5$ and $xy=3$,

(b) $x+y=5$ and $x^2+y^2=19$,

(c) $x^2+y^2=19$ and $xy=3$.

(1) Using the equality $x^2+y^2=(x+y)^2-\boxed{\text{F}}\,xy$, we see that

condition (b) gives $xy=\boxed{\text{G}}$,

condition (c) gives $x+y=\boxed{\text{H}}$ or $x+y=\boxed{\text{IJ}}$.

(2) For each of the following $\boxed{\text{K}}$ ~ $\boxed{\text{M}}$, choose the most appropriate answer from among the choices ⓪ ~ ③ below.

(i) (a) is $\boxed{\text{K}}$ for (b).

(ii) (b) is $\boxed{\text{L}}$ for (c).

(iii) (c) is $\boxed{\text{M}}$ for (a).

⓪ a necessary and sufficient condition

① a sufficient condition but not a necessary condition

② a necessary condition but not a sufficient condition

③ neither a necessary condition nor a sufficient condition

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This is the end of the questions for Part $\boxed{\text{I}}$. Leave the answer spaces $\boxed{\textbf{N}}$ ~ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{I}}$ blank.

$\boxed{\text{II}}$

Consider two straight lines

$$y = 1, \quad y = -1,$$

and the point A$(0, 3)$ in the xy-plane. Take a point P on the straight line $y = 1$ and a point Q on the straight line $y = -1$ such that

$$\angle \text{PAQ} = 90^\circ.$$

Let the two points P and Q move preserving the above conditions. We are to find the minimum value of the length of the line segment PQ.

First, denote the coordinates of P by $(\alpha, 1)$ and the coordinates of Q by $(\beta, -1)$. Then the condition $\angle \text{PAQ} = 90^\circ$ is reduced to the conditions $\alpha \neq 0, \beta \neq 0$ and

$$\alpha\beta = \boxed{\text{AB}}.$$

Since we know that α and β have opposite signs, let us assume that $\alpha < 0 < \beta$.

Then we have

$$\begin{aligned} \text{PQ}^2 &= (\beta - \alpha)^2 + \boxed{\text{C}} \\ &= \alpha^2 + \beta^2 + \boxed{\text{DE}} \\ &\geqq 2|\alpha\beta| + \boxed{\text{DE}} = \boxed{\text{FG}}. \end{aligned}$$

So we have

$$\text{PQ} \geqq \boxed{\text{H}}.$$

Hence, when

$$\alpha = \boxed{\text{IJ}}\sqrt{\boxed{\text{K}}} \quad \text{and} \quad \beta = \boxed{\text{L}}\sqrt{\boxed{\text{M}}},$$

PQ takes the minimum value $\boxed{\text{H}}$.

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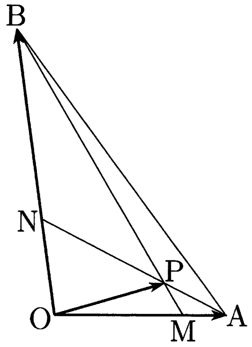
This is the end of the questions for Part $\boxed{\text{II}}$. Leave the answer spaces $\boxed{\textbf{N}}$ ～ $\boxed{\textbf{Z}}$ of Part $\boxed{\text{II}}$ blank.

III

Consider a triangle OAB. We denote by M the point dividing the side OA internally in the ratio 3 : 1 and denote by N the point dividing the side OB internally in the ratio 1 : 2. Also we denote by P the point of intersection of the line segment AN and the line segment BM.



(1) When the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are denoted by $\vec{a}$ and $\vec{b}$ respectively, we are to express the vector $\overrightarrow{OP}$ in terms of $\vec{a}$ and $\vec{b}$.

When we set

$$\mathrm{AP} : \mathrm{PN} = s : (1-s) \qquad (0 < s < 1)$$
$$\mathrm{BP} : \mathrm{PM} = t : (1-t) \qquad (0 < t < 1),$$

we have

$$\overrightarrow{OP} = (\boxed{\mathsf{A}} - s)\vec{a} + \frac{\boxed{\mathsf{B}}}{\boxed{\mathsf{C}}}s\vec{b}$$
$$= \frac{\boxed{\mathsf{D}}}{\boxed{\mathsf{E}}}t\vec{a} + (\boxed{\mathsf{F}} - t)\vec{b},$$

from which we obtain

$$s = \frac{\boxed{\mathsf{G}}}{\boxed{\mathsf{H}}}, \quad t = \frac{\boxed{\mathsf{I}}}{\boxed{\mathsf{J}}}.$$

Hence, $\overrightarrow{OP}$ can be expressed in terms of $\vec{a}$ and $\vec{b}$ as

$$\overrightarrow{OP} = \frac{\boxed{\mathsf{K}}}{\boxed{\mathsf{L}}}\vec{a} + \frac{\boxed{\mathsf{M}}}{\boxed{\mathsf{N}}}\vec{b}.$$

(This question is continued on the next page.)

(2) We are to look at the relation between the length of the line segment OP and the size of $\angle$AOB, when OA $= 6$ and OB $= 9$.

Let us denote the length of OP by ℓ. When we express ℓ^2 in terms of $\vec{a}\cdot\vec{b}$, we have

$$\ell^2 = \frac{\boxed{\text{O}}}{\boxed{\text{PQ}}}\,\vec{a}\cdot\vec{b} + \boxed{\text{RS}},$$

where $\vec{a}\cdot\vec{b}$ represents the inner product of $\vec{a}$ and $\vec{b}$.

Hence, for instance, if $\ell = 4$, then we have

$$\cos\angle\text{AOB} = \frac{\boxed{\text{TU}}}{\boxed{\text{V}}}.$$

When the size of $\angle$AOB varies, the range of the values which ℓ can take is

$$\boxed{\text{W}} < \ell < \boxed{\text{X}}.$$

This is the end of the questions for Part $\boxed{\text{III}}$. Leave the answer spaces $\boxed{\text{Y}}$, $\boxed{\text{Z}}$ of Part $\boxed{\text{III}}$ blank.

IV

Q 1 Let $f(x) = \log(4x - \log x)$, where log is the natural logarithm. We are to find a local extremum of $f(x)$ by using $f''(x)$.

For $\boxed{\text{K}}$ and $\boxed{\text{L}}$, choose the most appropriate answer from among the choices ⓪ ～ ⑥ below.

First of all, we have

$$f'(x) = \frac{\boxed{\text{A}} - \dfrac{\boxed{\text{B}}}{x}}{4x - \log x}$$

$$f''(x) = \frac{\dfrac{1}{x^{\boxed{\text{C}}}}(4x - \log x) - \left(\boxed{\text{A}} - \dfrac{\boxed{\text{B}}}{x}\right)^{\boxed{\text{D}}}}{(4x - \log x)^2},$$

which give

$$f'\left(\frac{\boxed{\text{E}}}{\boxed{\text{F}}}\right) = 0$$

$$f''\left(\frac{\boxed{\text{E}}}{\boxed{\text{F}}}\right) = \frac{\boxed{\text{GH}}}{\boxed{\text{I}} + \log \boxed{\text{J}}}.$$

Since

$$f''\left(\frac{\boxed{\text{E}}}{\boxed{\text{F}}}\right) \boxed{\text{K}}\ 0,$$

$f(x)$ has a $\boxed{\text{L}}$ at $x = \dfrac{\boxed{\text{E}}}{\boxed{\text{F}}}$, and this value is $\log(\boxed{\text{M}} + \log \boxed{\text{N}})$.

⓪ $=$ ① $>$ ② $\geqq$ ③ $<$ ④ $\leqq$ ⑤ local maximum ⑥ local minimum

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Q 2 Suppose that the curve $y = 2\cos 2x$ and the curve $y = 4\cos x + k$ have a common tangent at $x = a$ $\left(0 < a \leqq \frac{\pi}{2}\right)$.

(1) We set $f(x) = 2\cos 2x$ and $g(x) = 4\cos x + k$. Since we have assumed that $y = f(x)$ and $y = g(x)$ have a common tangent at $x = a$, we see that

$$f'(a) = g'(a), \quad f(a) = g(a).$$

Since $f'(a) = g'(a)$ and $0 < a \leqq \frac{\pi}{2}$, we have $a = \frac{\pi}{\boxed{\mathsf{O}}}$, and since $f(a) = g(a)$, we also have $k = -\boxed{\mathsf{P}}$.

Hence the coordinates of the tangent point are $\left(\frac{\pi}{\boxed{\mathsf{O}}}, -\boxed{\mathsf{Q}}\right)$, and the equation of the common tangent line is

$$y = -\boxed{\mathsf{R}}\sqrt{\boxed{\mathsf{S}}}\left(x - \frac{\pi}{\boxed{\mathsf{T}}}\right) - \boxed{\mathsf{U}}.$$

(2) We are to find the area S of the region bounded by these two curves over the range $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$.

Since both of these curves are symmetric with respect to the y-axis, by putting $b = \boxed{\mathsf{V}}$ and $c = \frac{\pi}{\boxed{\mathsf{O}}}$ we have

$$S = \boxed{\mathsf{W}}\int_b^c (2\cos 2x - 4\cos x - k)\,dx.$$

By calculating this, we obtain

$$S = \boxed{\mathsf{X}}\,\pi - \boxed{\mathsf{Y}}\sqrt{\boxed{\mathsf{Z}}}.$$

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This is the end of the questions for Part $\boxed{\text{IV}}$.

This is the end of the questions for Course 2. Leave the answer spaces for Part $\boxed{\text{V}}$ blank.

Please check once more that you have properly marked your course number as "Course 2" on your answer sheet.

Do not take this question booklet out of the room.